Linearteam tech paper

The analysis of fourth-order state variable filter and it's application to Linkwitz-Riley filters

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1. Introduction

The Linkwitz-Riley-filter is considered to be one of the best analog crossover filter functions. It has many desirable properties, such as constant phase between outputs. Resulting summed transfer function is of all-pass-type. It exhibits some phase distortion, but flat amplitude response, assuming that no delays are introduced after the filter before the summation point.

In this paper, I'll derive necessary equations to design Linkwitz-Riley filter or any other combination of two identical second-order filter sections by using state-variable filter. State variable filter is nice, if you need identical high-pass and low-pass sections simultaneously, because it implements both of them. Furthermore, they have identical cut-off-frequencies, so requirement for precision components is reduced.

Originally, I saw this type of circuit in Finnish book, called "Rakenna HIFI-laitteita". It describes this type of filter, by using state-variable filter, but leaves reader no clue how the feedback resistor networks should be designed to have required Q and K. It only states that "the Q-value is set by designing feedback loops appropriately". Either the author has no idea how it is done, or is unwilling to reveal the information. This paper should fill the gap. Furthermore, I can't stand circuits which I have no clue how the design is made.

My derivation follows guidelines given in [1]. Although actual circuit here is a bit different, analysis is still basically the same. There are some mathematics involved here, but that is a bit unavoidable during analysis of circuits like this.

2. Fourth-order Linkwitz-Riley (LR4) transfer function

Generalized second order high-pass transfer function is defined to be

$$H(s) = \frac{Ks^2}{s^2 + s\left(\frac{w_0}{Q}\right) + w_0^{\,2}} \tag{1}$$

where Q is the quality factor, i.e. absolute gain at natural angular frequency ω_0 and K is gain at high frequencies, $\omega >> \omega_0$.

Generally, the fourth order transfer function is

$$H_{HP4}(s) = H_{HP2_1}(s) \cdot H_{HP2_2}(s) = \left(\frac{K_1 s^2}{s^2 + s\left(\frac{w_{01}}{Q_1}\right) + w_{01}^{\,2}}\right) \cdot \left(\frac{K_2 s^2}{s^2 + s\left(\frac{w_{02}}{Q_2}\right) + w_{02}^{\,2}}\right) \tag{2}$$

Multiplying these two parts together we get for transfer function

$$H_{HP4}(s)=\frac{K_1K_2s^4}{s^4+s^3\left(\frac{w_{01}}{Q_1}+\frac{w_{02}}{Q_2}\right)+s^2\left(w_{01}{}^2+\frac{w_{01}w_{02}}{Q_1Q_2}+w_{02}{}^2\right)+s\left(\frac{w_{01}{}^2w_{02}}{Q_1}+\frac{w_{01}w_{02}{}^2}{Q_2}\right)+w_{01}{}^2w_{02}{}^2} \quad (3)$$

LR4 transfer function can be formed by composing identical two second order butterworth filters, which have Q of $\frac{1}{\sqrt{2}}$, same natural frequency and usually some desired gain, so we make following assignments:

$$w_{01}=w_{02}=w_0$$

$$Q_1=Q_2=Q$$

$$K_1=K_2=K$$

With these simplifications, the transfer function (x) is reduced to

$$H_{HP4}(s)=\frac{KKs^4}{s^4+s^3\left(\frac{w_0}{Q}+\frac{w_0}{Q}\right)+s^2\left(w_0{}^2+\frac{w_0w_0}{QQ}+w_0{}^2\right)+s\left(\frac{w_0{}^2w_0}{Q}+\frac{w_0w_0{}^2}{Q}\right)+w_0{}^2w_0{}^2} \quad (4)$$

and further to

$$H_{HP4}(s)=\frac{K^2s^4}{s^4+s^3\left(\frac{2w_0}{Q}\right)+s^2\left(\frac{w_0{}^2\left(1+2Q^2\right)}{Q^2}\right)+s\left(\frac{2w_0{}^3}{Q}\right)+w_0{}^4} \quad (5)$$

3. Transfer function of state-variable architecture

To fit the LR4 transfer function, we use basic definition of transfer function:

$$H_{HP4}(s)=\frac{\mathrm{U_{hp}}(s)}{\mathrm{U_{in}}(s)}=\frac{K^2s^4}{s^4+s^3\left(\frac{2w_0}{Q}\right)+s^2\left(\frac{w_0{}^2\left(1+2Q^2\right)}{Q^2}\right)+s\left(\frac{2w_0{}^3}{Q}\right)+w_0{}^4} \quad (6)$$

To obtain desired form, we’ll cross-multiply eq. (7), and divide both sides by s^4:

$$\mathrm{U_{hp}}(s)+\mathrm{U_{hp}}(s)\left(\frac{w_0}{s}\frac{2}{Q}\right)+\mathrm{U_{hp}}(s)\left(\frac{w_0{}^2}{s^2}\frac{\left(1+2Q^2\right)}{Q^2}\right)+\mathrm{U_{hp}}(s)\left(\frac{w_0{}^3}{s^3}\frac{2}{Q}\right)+\mathrm{U_{hp}}(s)\frac{w_0{}^4}{s^4}=K^2\mathrm{U_{in}}(s) \quad (7)$$

Preceding equation somewhat gives us a hint, how state-variable filter is formed:

First, we form high-pass filtered signal, and applying successively integrators, we get to lowpass function after fourth integrator. Problem is, however, how we obtain the high-pass filtered signal in the first place. By arranging the equation as

$$\mathrm{U}_{\mathrm{hp}}(s) = K^2 \mathrm{U}_{\mathrm{in}}(s) - \frac{2}{Q}\left(\frac{w_0}{s}\mathrm{U}_{\mathrm{hp}}(s)\right) - \frac{(1+2Q^2)}{Q^2}\left(\frac{{w_0}^2}{s^2}\mathrm{U}_{\mathrm{hp}}(s)\right) - \frac{2}{Q}\left(\frac{{w_0}^3}{s^3}\mathrm{U}_{\mathrm{hp}}(s)\right) - \frac{{w_0}^4}{s^4}\mathrm{U}_{\mathrm{hp}}(s) \qquad (8)$$

We see, that $\mathrm{U}_{\mathrm{hp}}(s)$ is obtainable by using weighted summer circuit from intermediate outputs. Also, with this arrangement, connection to integrator implementation is particularly strong.

Realization of this transfer function is shown in figure 1 below:

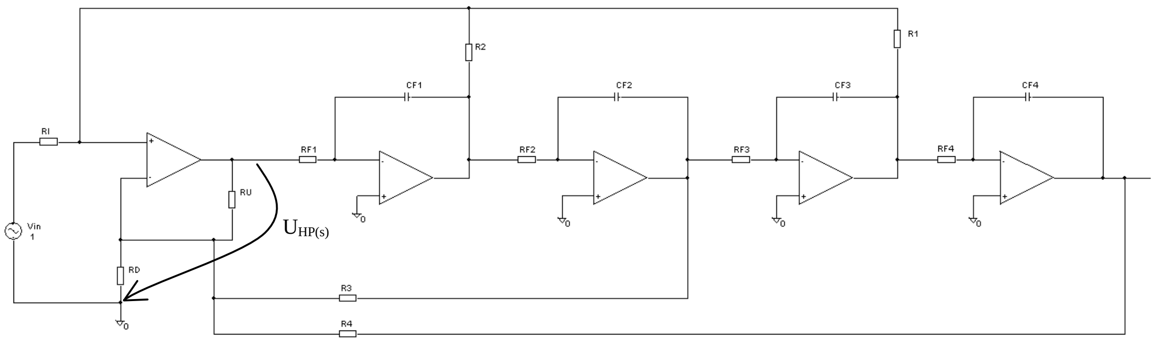


Figure 1. Schematics of fourth order state-variable filter realization.

This circuit is most conveniently analyzed with superposition. That means that we consider only one source per time, and rest are thought to be zero with zero output impedance.

With component references in figure x, we get following expression for voltage $U_{HP}(s)$ at output of the summer:

$$\begin{aligned} U_{HP}(s) = &\left(\frac{R_1 \| R_2}{R_I + R_1 \| R_2}\right)\left(1+\frac{R_U}{R_D \| R_3 \| R_4}\right) U_{IN}(s) + \left(\frac{R_1 \| R_I}{R_2 + R_1 \| R_I}\right)\left(1+\frac{R_U}{R_D \| R_3 \| R_4}\right)\left(-\frac{w_0}{s}\right) U_{HP}(s) \\ &-\left(\frac{R_U}{R_3}\right)\left(\frac{{w_0}^2}{s^2}\right) U_{HP}(s) + \left(\frac{R_2 \| R_I}{R_1 + R_2 \| R_I}\right)\left(1+\frac{R_U}{R_D \| R_3 \| R_4}\right)\left(-\frac{{w_0}^3}{s^3}\right) U_{HP}(s) - \left(\frac{R_U}{R_4}\right)\left(\frac{{w_0}^4}{s^4}\right) U_{HP}(s) \end{aligned} \qquad (9)$$

Each integrator's time constant is $R_F C_F$, which is chosen to be

$$R_F C_F = \frac{1}{w_0} = \frac{1}{2pf_0} \qquad (10)$$

4. Circuit synthesis

We just derived expression for U_{HP}. Now we just equate equations (x) and (y), and get following equations:

$$\begin{cases} K^2 = \left(\dfrac{R_1 \| R_2}{R_I + R_1 \| R_2}\right)\left(1 + \dfrac{R_U}{R_D \| R_3 \| R_4}\right) = \left(\dfrac{1}{1 + \dfrac{R_I}{R_2} + \dfrac{R_I}{R_1}}\right)\left(1 + \dfrac{R_U}{R_D} + \dfrac{R_U}{R_4} + \dfrac{R_U}{R_3}\right) \\ \dfrac{2}{Q} = \left(\dfrac{R_1 \| R_I}{R_2 + R_1 \| R_I}\right)\left(1 + \dfrac{R_U}{R_D \| R_3 \| R_4}\right) = \left(\dfrac{1}{1 + \dfrac{R_1}{R_2} + \dfrac{R_2}{R_I}}\right)\left(1 + \dfrac{R_U}{R_D} + \dfrac{R_U}{R_4} + \dfrac{R_U}{R_3}\right) \\ \dfrac{\left(1 + 2Q^2\right)}{Q^2} = \dfrac{R_U}{R_3} \\ \dfrac{2}{Q} = \left(\dfrac{R_2 \| R_I}{R_1 + R_2 \| R_I}\right)\left(1 + \dfrac{R_U}{R_D \| R_3 \| R_4}\right) = \left(\dfrac{1}{1 + \dfrac{R_1}{R_2} + \dfrac{R_1}{R_I}}\right)\left(1 + \dfrac{R_U}{R_D} + \dfrac{R_U}{R_4} + \dfrac{R_U}{R_3}\right) \\ 1 = \dfrac{R_U}{R_4} \end{cases} \quad (11)$$

In order to simplify these requirements, we can make following conclusions from previous equations: Resistor ratio $\frac{R_U}{R_4}$ equals one, so we can combine it with ratio $\frac{R_U}{R_3}$ so that $\frac{R_4}{R_3} = \frac{\left(1 + 2Q^2\right)}{Q^2}$. That leaves us still four unknown ratios and three equations to work with, so we can select one ratio arbitarily. In previously mentioned source, R_1 and R_2 are equally valued, so we can make same selection. I'll mark $R_1=R_2=R_{12}$. Previous set of equations is reduced to

$$\begin{cases} K^2 = \left(\dfrac{1}{1 + \dfrac{R_I}{R_{12}} + \dfrac{R_I}{R_{12}}} \right) \left(2 + \dfrac{R_U}{R_D} + \dfrac{R_4}{R_3} \right) = \left(\dfrac{1}{1 + \dfrac{2R_I}{R_{12}}} \right) \left(2 + \dfrac{R_U}{R_D} + \dfrac{R_4}{R_3} \right) \\ \dfrac{2}{Q} = \left(\dfrac{1}{2 + \dfrac{R_{12}}{R_I}} \right) \left(2 + \dfrac{R_U}{R_D} + \dfrac{R_4}{R_3} \right) \\ \dfrac{\left(1 + 2Q^2\right)}{Q^2} = \dfrac{R_4}{R_3} \end{cases} \tag{12}$$

By making following substitutions to simplify rest of the derivation of synthesis eqs:

$$A = \frac{R_{12}}{R_I},\ B = \frac{R_U}{R_D},\ C = \frac{R_4}{R_3}$$

we can denote above eqs as

$$\begin{cases} K^2 = \dfrac{(2 + B + C)}{1 + \dfrac{2}{A}} \\ \dfrac{2}{Q} = \dfrac{(2 + B + C)}{2 + A} \\ \dfrac{\left(1 + 2Q^2\right)}{Q^2} = C \end{cases} \tag{13}$$

By combining first two eqs, we can solve resistor ratios A, B and C:

$$2A^2 + \left(4 - QK^2\right)A - 2QK^2 = 0 \Leftrightarrow A = \frac{QK^2 - 4 + \sqrt{\left(4 - QK^2\right)^2 + 16QK^2}}{4} \tag{14}$$

$$C = \frac{\left(1 + 2Q^2\right)}{Q^2} \tag{15}$$

$$B = \frac{4 + 2A}{Q} - 2 - C \tag{16}$$

5. Examples

Example 1:

As an example, let's design LR4 filter, what has following specifications: f_c=3.5 kHz and K^2=2 (6.02 dB).

First we compute constant A:

$$A = \frac{QK^2 - 4 + \sqrt{\left(4 - QK^2\right)^2 + 16QK^2}}{4} = \frac{\frac{1}{\sqrt{2}} \cdot 2 - 4\sqrt{\left(4 - \frac{1}{\sqrt{2}}2\right)^2 + 16\frac{1}{\sqrt{2}}2}}{4} = \underline{\frac{1}{\sqrt{2}} \approx 0,7071} \qquad (17)$$

Then C:

$$C = \frac{\left(1 + 2Q^2\right)}{Q^2} = \frac{\left(1 + 2\left(\frac{1}{\sqrt{2}}\right)^2\right)}{\left(\frac{1}{\sqrt{2}}\right)^2} = \underline{4} \qquad (18)$$

And finally B:

$$B = \frac{4 + 2A}{Q} - 2 - C = \frac{4 + 2\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} - 2 - 4 = \underline{\sqrt{32} - 4 \approx 1,657} \qquad (19)$$

Now when we revert back to original values, we can calculate resistor values:

$$A = \frac{R_{12}}{R_I},\ B = \frac{R_U}{R_D},\ C = \frac{R_4}{R_3}$$

Choosing R_I as 10 kΩ, we get R_1 and R_2 :

$$R_{12} = R_1 = R_2 = R_I A = 10\ \text{kO}\frac{1}{\sqrt{2}} \approx \underline{7,071\,\text{kO}} \qquad (20)$$

Then, by choosing for R_4 and R_U also as 10 kΩ, we get R_D:

$$R_D = \frac{R_U}{B} = \frac{10\ \text{kO}}{\sqrt{32} - 4} \approx \underline{6,036\ \text{kO}} \qquad (21)$$

and R3:

$$R_3 = \frac{R_4}{C} = \frac{10\,\text{kO}}{4} \approx \underline{2,5\,\text{kO}} \qquad (22)$$

C_F is chosen to be 10 nF, and R_F calculated:

$$R_F = \frac{1}{2pC_F f_0} = \frac{1}{2p10\cdot10^{-9}3.5\cdot10^3} \approx \underline{4547{,}28\Omega} \qquad (23)$$

Finalized circuit schematics looks like this:

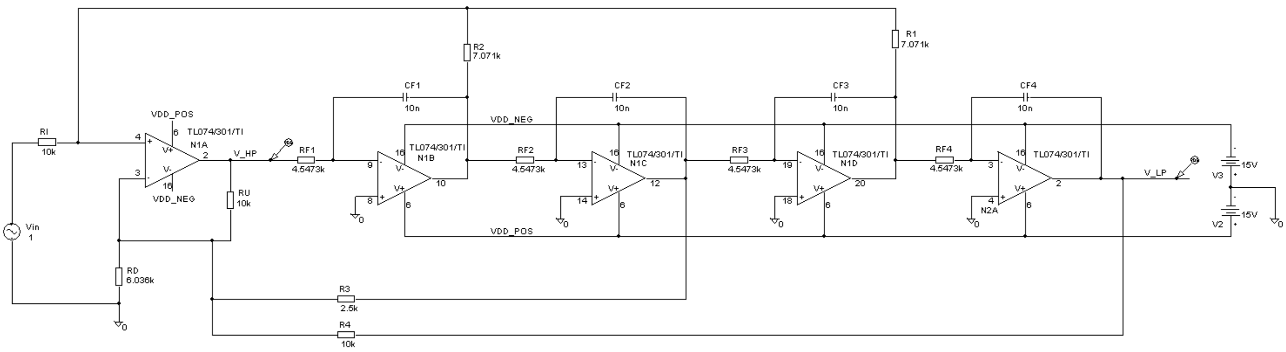


Figure 2. Completed LR4 filter realization schematics with (Q=0.707, K^2=2, f_c=3.5 kHz).

Below is a output from simulation using previously calculated resistor values with PSPICE:

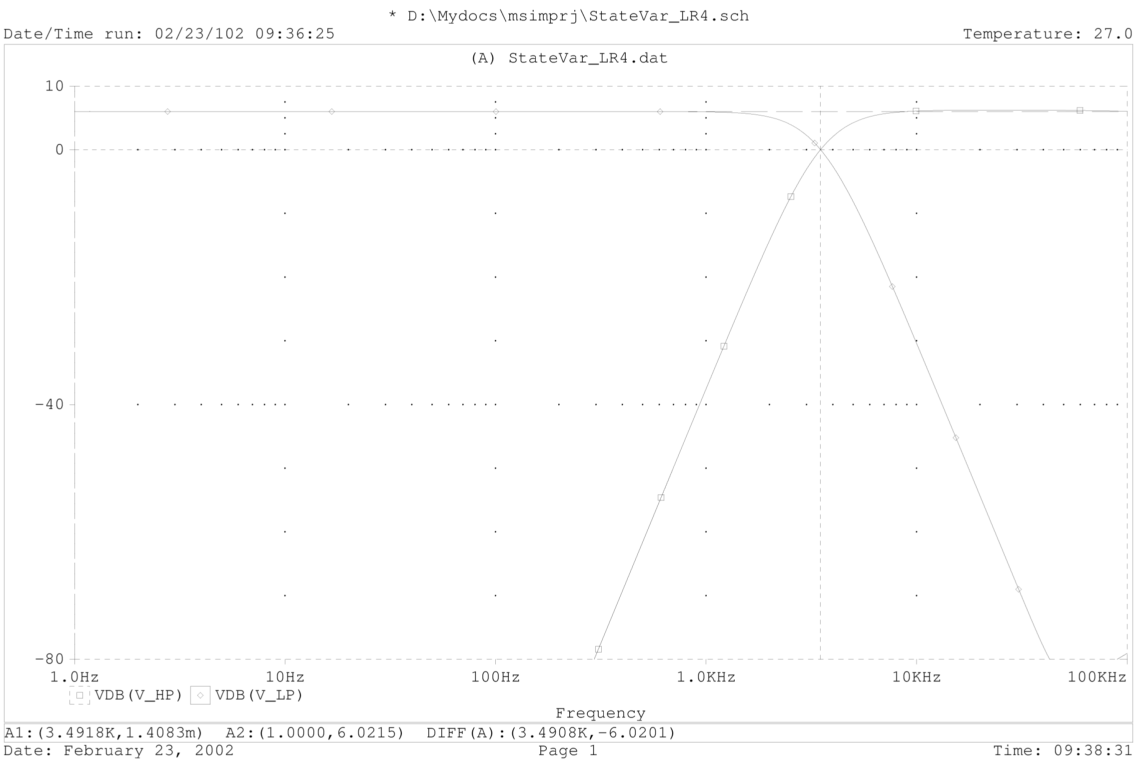


Figure 3. Simulated frequency response of filter in fig.2.

Example 2:

Now, design again a LR4 Filter, which has total gain K^2 as 1, and cut-off frequency f_c of 50 Hz (same case as in [2]):

First we compute constant A:

$$A = \frac{QK^2 - 4 + \sqrt{\left(4 - QK^2\right)^2 + 16QK^2}}{4} = \frac{\frac{1}{\sqrt{2}} \cdot 1 + 4\sqrt{\left(4 - \frac{1}{\sqrt{2}}1\right)^2 + 16\frac{1}{\sqrt{2}}1}}{4} \approx \underline{0{,}3536} \tag{24}$$

Then C:

$$C = \frac{\left(1 + 2Q^2\right)}{Q^2} = \frac{\left(1 + 2\left(\frac{1}{\sqrt{2}}\right)^2\right)}{\left(\frac{1}{\sqrt{2}}\right)^2} = \underline{4} \tag{25}$$

And finally B:

$$B = \frac{4 + 2A}{Q} - 2 - C = \frac{4 + 2 \cdot 0{,}3536}{\frac{1}{\sqrt{2}}} - 2 - 4 \approx \underline{0{,}6569} \tag{26}$$

Now when we revert back to original values, we can calculate resistor values:

$$A = \frac{R_{12}}{R_I},\ B = \frac{R_U}{R_D},\ C = \frac{R_4}{R_3}$$

Choosing R_I as 10 kΩ, we get R_1 and R_2 :

$$R_{12} = R_1 = R_2 = R_I A = 10\ \text{kO} \cdot 0{,}3536 \approx \underline{3{,}536\ \text{kO}} \tag{27}$$

Then, by choosing for R_4 and R_U also as 10 kΩ, we get R_D:

$$R_D = \frac{R_U}{B} = \frac{10\,\text{kO}}{0{,}6569} \approx \underline{15{,}22\,\text{kO}} \tag{28}$$

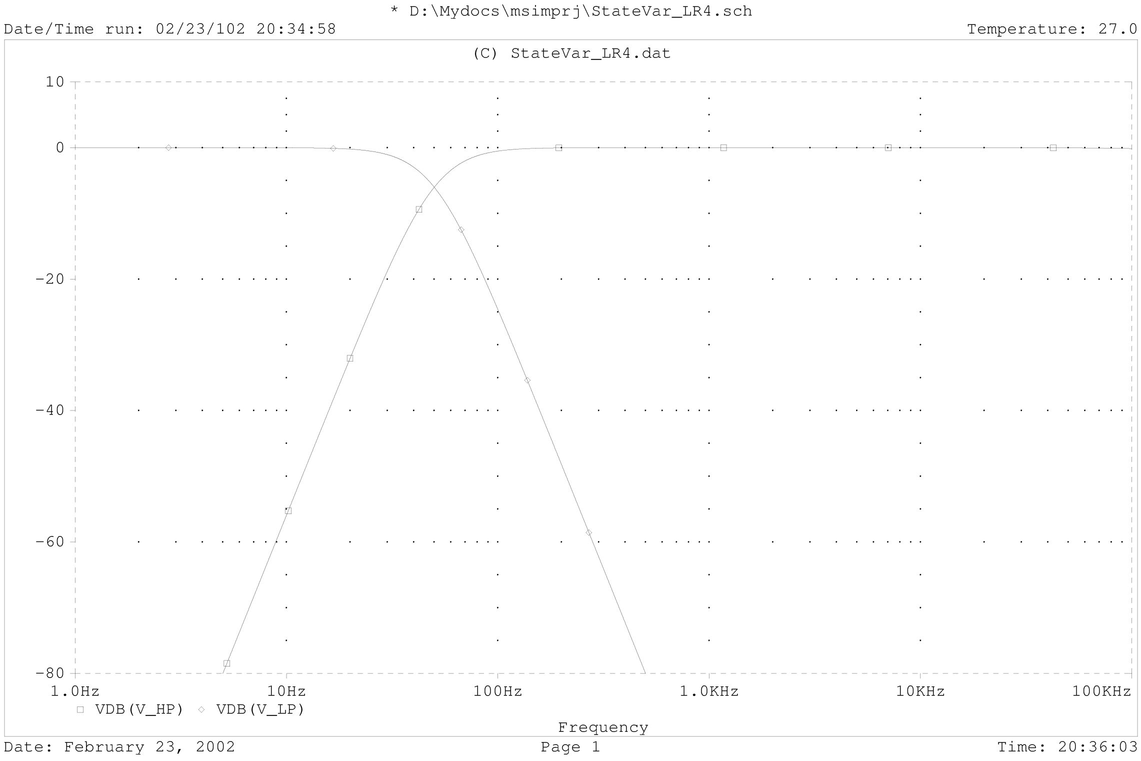
and R3:

$$R_3 = \frac{R_4}{C} = \frac{10\,\text{kO}}{4} \approx \underline{2{,}5\,\text{kO}} \tag{29}$$

C_F is chosen to be 100 nF, and R_F calculated:

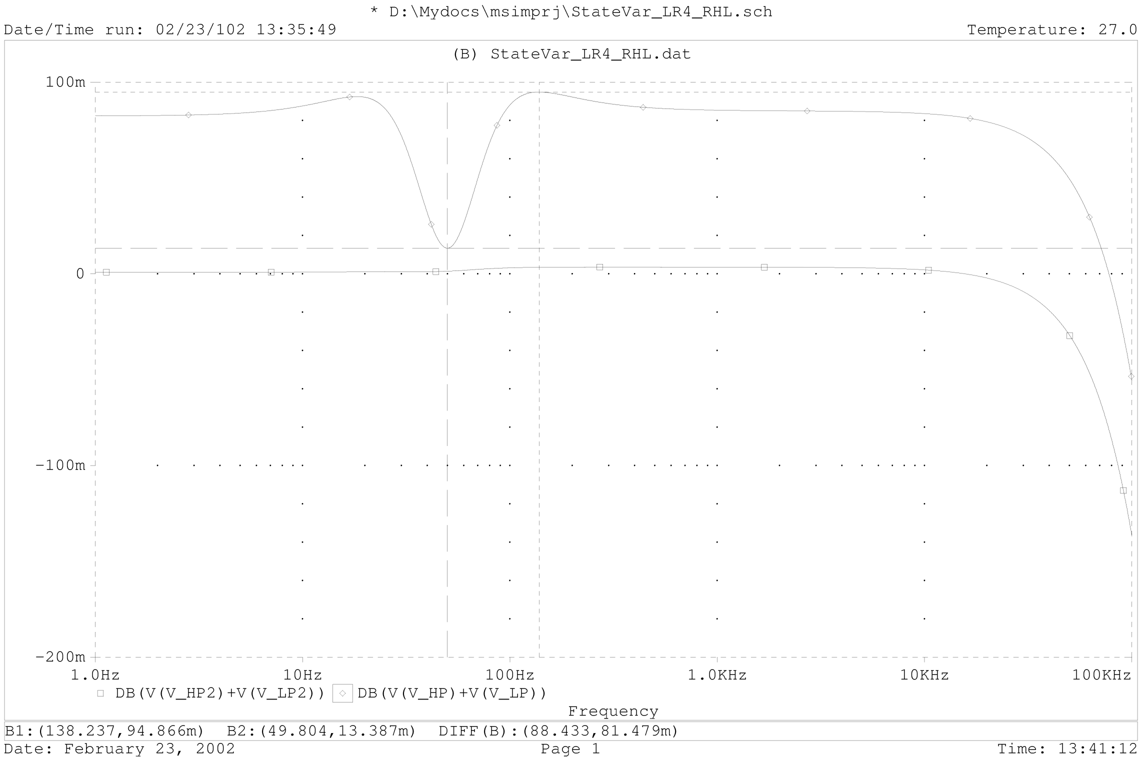
$$R_F = \frac{1}{2pC_F f_0} = \frac{1}{2p100 \cdot 10^{-9} 50} \approx \underline{31{,}83\text{kO}} \qquad (30)$$

Again simulation agrees well:



6. Realization accuracy

State-variable architecture is somewhat sensitive to opamp finite gain-bandwith-product. Here is plot of summed frequency response from high and lowpass outputs. Simulation was here on example 2, because it lets you compare with resistor values given in [2]. Ideally, it should be completely flat. Using resistor values from [2], there is some more deviation to response:



Upper summed frequency response shows 0.08 dB deviation from ideal behaviour using otherwise identical parts. Below it, is a trace using calculated ideal component values.

7. References

[1] Sedra-Smith, Microelectronic Circuits

[2] Rakenna HIFI-laitteita, Helsinki Media