Equalization of the closed box

Janne Ahonen

29.1.2003

TABLE OF CONTENTS

1 INTRODUCTION

Equalization on subwoofers is commonly very little covered subject. This document is written to enlighten some aspects of this area of box design. During the development of mathematical models for WinISD, I had gained some knowledge about how to systematically add zeros/poles to system in order to achieve desired response. This document contains some ”advanced” mathematics (well, not very advanced, but to get maximum benefit from this you should be at least familiar with complex numbers. It is also useful to understand the concept behind so called transfer functions.). I have also found out that local audio magazine, ”HIFI-lehti” seems to be unwilling to disclose any detailed information on subject (at least that way, that it could be used to do more research on the subject). There are two things which annoy me very much:

1) Equations are given ”pre-adjusted” in specific situation, so there is no possibility to further fiddle with it, and,

2) Equations are scaled so that variables aren’t given as SI-baseunits, as they should.

I’ll promise I won’t do that.

Warning! If you think that it is difficult to do conversion between metric system and imperial units, then please don’t bother to read this document :) I have as much as possible given much thought to make this as ”JAES-quality” text, so it might not be suitable for beginners.

Please feel free to send any corrections, additions, feedback, suggestions about this document.

Contact info:

Janne Ahonen
Aittolammentie 3 as 7
70780 Kuopio
Finland

e-mail: janne@linearteam.org.

2 BASIC THINGS

I'll briefly explain basics for some concepts behind equalizer design.

2.1 Transfer functions

Transfer functions are based upon integral conversion called 'Laplace-transform'. Basically, it transforms time domain function to 's-domain', where time is no longer a variable.

Fortunately, it is not usually necessary to perform actual Laplace transform to find the transfer function for the circuit. Instead, it is usually done by applying the known voltage divider formula in AC-domain.

Variable in s-domain is s (surprise!), which is defined as $s = jw + s$. Where $j = \sqrt{-1}$ or imaginary unit. The variable ω is the angular frequency which is related to normal frequency by

$$w = 2pf \tag{2.1}$$

Transfer functions are commonly rational functions, where denominator and numerator are polynomials of s. Roots of denominator polynomial are called as 'poles' and roots of numerator polynomial are called as 'zeros', respectively. This paper deals exclusively with second order transfer functions, because closed box speaker can be interpreted as acoustic second order highpass filter, which has Q and natural angular frequency w_n .

Second order lowpass transfer function has following form:

$$H(s) = \frac{K \cdot w_n^{\;2}}{s^2 + 2xw_n s + w_n^{\;2}} = \frac{1}{\frac{1}{Kw_n^{\;2}}s^2 + \frac{2x}{Kw_n}s + \frac{1}{K}} \tag{2.2}$$

where ω_n is natural frequency of the system in radians/sec, ξ is the damping factor and K is the gain. Q is related to ξ by relation

$$Q = \frac{1}{2x} \tag{2.6}$$

Then, we can write (2.2) as

$$H(s) = \frac{K \cdot w_n^{\;2}}{s^2 + 2\frac{1}{2Q}w_n s + w_n^{\;2}} = \frac{K \cdot w_n^{\;2}}{s^2 + \frac{w_n}{Q}s + w_n^{\;2}} = \frac{1}{\frac{1}{Kw_n^{\;2}}s^2 + \frac{1}{KQw_n}s + \frac{1}{K}} \tag{2.7}$$

Equation (2.2) can be transformed into high-pass form by using LP→HP frequency transformation. For that, we must assume that w_n equals one. The most commonly used frequency transformations are :

Lowpass to lowpass (LP→LP):

$$s = \frac{s}{w_n} \tag{2.8}$$

Lowpass to highpass (LP→HP):

$$s = \frac{w_n}{s} \tag{2.9}$$

Lowpass to bandpass (LP→BP):

$$s = \frac{s^2 + w_n{}^2}{Bs} \tag{2.10}$$

Lowpass to band-reject (LP→BR):

$$s = \frac{Bs}{s^2 + w_n{}^2} \tag{2.11}$$

Now, let's apply LP→HP frequency transformation to (2.2)

$$H(s) = \frac{K \cdot w_n{}^2}{s^2 + \frac{w_n}{Q} s + w_n{}^2} \Big\| w_n = 1 \tag{2.12}$$

and then

$$H(s) = \frac{K}{s^2 + \frac{1}{Q} s + 1} \Bigg\| s = \frac{w_n}{s} \tag{2.13}$$

Which gives

$$\begin{aligned} H(s) &= \frac{K}{\left(\frac{w_n}{s}\right)^2 + \frac{1}{Q}\frac{w_n}{s} + 1} = \frac{KQ \cdot s^2}{Qs^2 + w_n s + Qw_n{}^2} \\ &= \frac{Ks^2}{s^2 + \frac{w_n}{Q} s + w_n{}^2} = \frac{s^2}{\frac{1}{K} s^2 + \frac{w_n}{KQ} s + \frac{w_n{}^2}{K}} \end{aligned} \tag{2.14}$$

Or, by using damping factor ξ,

$$H(s) = \frac{K \frac{1}{2x} \cdot s^2}{\frac{1}{2x} s^2 + w_n s + \frac{1}{2x} w_n{}^2} = \frac{K \cdot s^2}{s^2 + 2w_n xs + w_n{}^2} = \frac{s^2}{\frac{1}{K} s^2 + \frac{2xw_n}{K} s + \frac{w_n{}^2}{K}} \tag{2.15}$$

Denoting the lowpass transfer function denominator coefficients $a_2 \ldots a_0$ we can express second order transfer function as

$$H(s) = \frac{1}{a_2 s^2 + a_1 s + a_0} \tag{2.16}$$

Coefficients $a_2 \ldots a_0$ can then be interpreted as follows:

$$\begin{cases} a_2 = \dfrac{1}{K w_n{}^2} \\ a_1 = \dfrac{2x}{K w_n} \\ a_0 = \dfrac{1}{K} \end{cases} \tag{2.17}$$

Solving these for K, ω_n, and ξ, we obtain following:

$$\begin{cases} K = \dfrac{1}{a_0} \\ w_n = \sqrt{\dfrac{a_0}{a_2}} \\ x = \dfrac{a_1}{2\sqrt{a_0 a_2}} \end{cases} \tag{2.18}$$

So for Q, we get

$$Q = \frac{1}{2\dfrac{a_1}{2\sqrt{a_0 a_2}}} = \frac{\sqrt{a_0 a_2}}{a_1} \tag{2.19}$$

2.2 Types of second order transfer functions

There are basically three types of second order transfer functions:

- Overdamped $Q<0.5$
- Critically damped $Q=0.5$
- Underdamped, $Q>0.5$

Above types differ only by types of roots of denominator. Overdamped systems have all roots on real axis. If we solve poles for transfer function H(s), then second order transfer function for overdamped system can be expressed as follows:

$$H(s)=\frac{1}{(s-p_1)(s-p_2)}=\frac{1}{\left(s\frac{1}{-p_1}+1\right)\left(s\frac{1}{-p_2}+1\right)} \tag{2.20}$$

where p_1 and p_2 are the poles.

Because this document involves with closed box, which is a second-order system, following relation is therefore useful:

For system, that has $Q \leq 0{,}5$ poles p_1 and p_2 are as follows:

$$H(s)=\frac{1}{(st_1+1)\cdot(st_2+1)}=\frac{1}{s^2\cdot(t_1t_2)+s(t_1+t_2)+1} \tag{2.21}$$

Note that closed box has double zeros at origin, but response is set by poles (zeros ignored).

Where $t_1=\frac{1}{-p_1}$ and $t_2=\frac{1}{-p_2}$. This notation is corresponding to RC-circuit, where time constant is expressed as $t = RC$. When dealing with real poles it is convenient to think these as two series connected independent RC-circuits, where time constants are determined from poles.

Here, coefficients for (2.9) are:

$$\begin{cases} a_2 = t_1t_2 \\ a_1 = t_1 + t_2 \\ a_0 = 1 \end{cases} \tag{2.22}$$

By using (2.5) and (2.7) we get expressions for Q_{tc} and f_{sc} (gain K is normally set to 1):

$$\begin{cases} Q_{tc} = \dfrac{\sqrt{t_{p1}t_{p2}}}{t_{p1}+t_{p2}} = \dfrac{\sqrt{f_{p1}f_{p2}}}{f_{p1}+f_{p2}} \\ f_{sc} = \dfrac{\sqrt{\dfrac{1}{t_{p1}t_{p2}}}}{2p} = \dfrac{1}{2p\sqrt{t_{p1}t_{p2}}} = \sqrt{f_{p1}f_{p2}} \end{cases} \tag{2.23}$$

When designing single-pole compensators, it is useful to solve f_{p1} and f_{p2} from Q_{tc} and f_{sc}. Using (2.23) we obtain

$$\begin{cases} f_{p1} = \dfrac{f_{sc}-\sqrt{f_{sc}^{\ 2}-4f_{sc}^{\ 2}Q_{tc}^{\ 2}}}{2Q_{tc}} = f_{sc}\dfrac{1-\sqrt{1-4Q_{tc}^{\ 2}}}{2Q_{tc}} \\ f_{p2} = \dfrac{f_{sc}+\sqrt{f_{sc}^{\ 2}-4f_{sc}^{\ 2}Q_{tc}^{\ 2}}}{2Q_{tc}} = f_{sc}\dfrac{1+\sqrt{1-4Q_{tc}^{\ 2}}}{2Q_{tc}} \end{cases}$$

It is also useful to derive an magnitude function of single real pole of high-pass function:

$$H(s) = \frac{s}{s-p} \tag{2.24}$$

Multiplying this with complex conjugate of the transfer function, we get

$$H(s)H(s)^* = |H(s)|^2 = \frac{s}{(s-p)} \cdot \frac{-s}{(-s-p)} = \frac{-s^2}{p^2-s^2} = \frac{-(jw)^2}{p^2-(jw)^2} = \frac{w^2}{p^2+w^2} \tag{2.25}$$

So magnitude function of single pole is therefore

$$|H(s)| = \sqrt{\frac{w^2}{p^2+w^2}} = \frac{w^2}{\sqrt{1+\left(\frac{w}{p}\right)^2}} = \sqrt{\frac{w^2}{1+(wt)^2}} = \frac{w}{\sqrt{1+(wt)^2}} \tag{2.26}$$

Critically damped system is special case of underdamped systems. It has both poles on same location on real axis. Underdamped system has complex conjugate pole pair. Complex poles of real system occur always as complex conjugate pair, because otherwise system would have to be complex. Generalized second order high pass magnitude function is

$$\begin{aligned}|H(s)| &= \sqrt{H(s)H^*(s)} = \sqrt{\frac{b_2 s^2}{a_2 s^2 + a_1 s + a_0} \cdot \frac{b_2 s^2}{a_2 s^2 - a_1 s + a_0}} \\ &= \frac{b_2 w^2}{\sqrt{a_2{}^2 w^4 + \left(a_1{}^2 - 2a_2 a_0\right)w^2 + a_0{}^2}}\end{aligned} \tag{2.27}$$

By substituting

$$\begin{cases} b_2 = K \\ a_2 = 1 \\ a_1 = \dfrac{w_n}{Q} \\ a_0 = w_n{}^2 \end{cases} \tag{2.28}$$

to (2.15) we get

$$|H(s)| = \frac{Kw^2}{\sqrt{w^4 + \left(\left(\frac{w_n}{Q}\right)^2 - 2w_n\right)w^2 + w_n{}^4}} \tag{2.29}$$

which can be even further simplified.

2.3 The s-plane

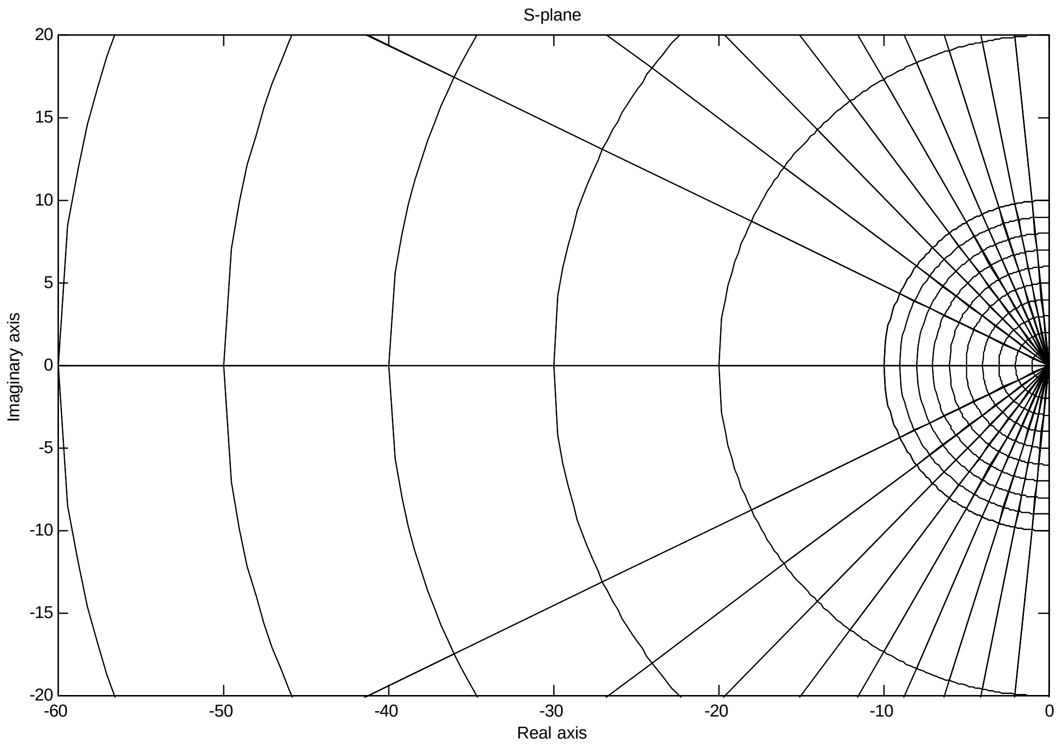

Figure 1. Example of the s-plane.

Circles represent points, where pole has constant natural frequency. When pole travels along this line, its so called damping factor is changed from 1 (pole at real axis) to 0 (pole at imaginary axis).

Formally, the damping factor is defined to be

$$x = \cos(y) \tag{2.14}$$

where ψ is the angle between real axis and the pole, and the natural frequency of the pole in radians/sec is defined to be

$$w_n = |s| \tag{2.15}$$

Lines originating from the origin represent points, where pole has constant damping factor and its natural frequency changes.

Interesting fact is that Q-factor tells what is the gain at the natural frequency of second-order system.

Preceding statement is illustrated in following figures:

A second-order high-pass transfer function's magnitude, whose natural frequency was set to 20 Hz and Q_{tc} varied from 0.3 to 2.0 is plotted below:

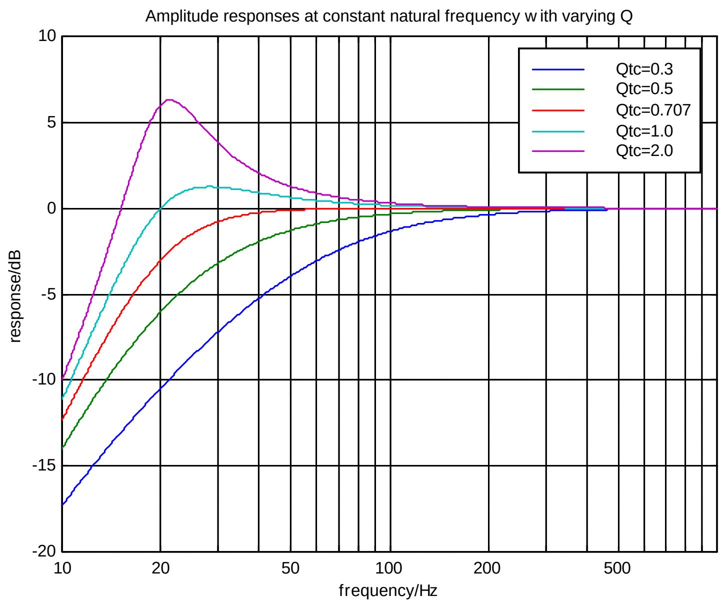

Figure 2. Amplitude response of various transfer functions with different Q-values.

From Figure 2. it is really evident that Q really is the gain at natural frequency. Transfer function with Q_{tc}=1.0, the response crosses exactly 0 dB at 20 Hz. Let's see pole-zero map for same transfer functions:

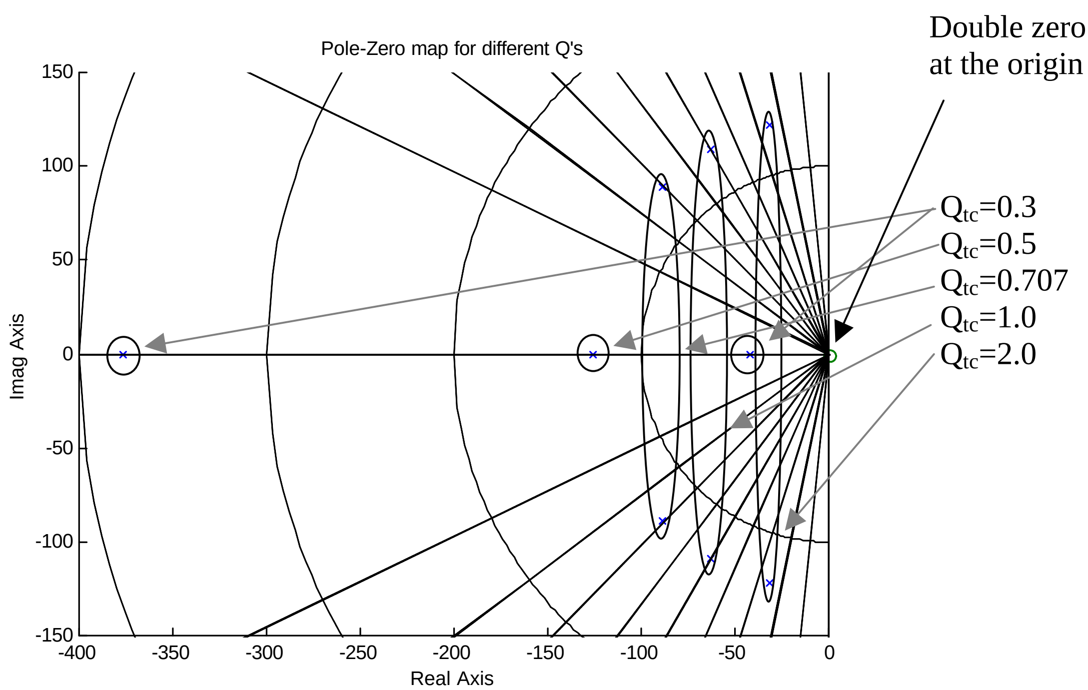

Figure 3. Pole-zero map for previous transfer functions

When Q_{tc}=0,3 the poles are located at real axis. With Q_{tc}=0,5, the poles are merged to ”one” pole (both poles at same location). With Q_{tc}=0,707, the poles are complex, and their angle between real axis and the poles is 45°. As Q_{tc} continues to increase, the poles move along constant ω_n circle towards the imaginary axis.

2.4 Determining transfer functions from passive networks

When determining component impedances in s-domain σ is set to zero. So we actually get following impedances in s-domain:

Table 1. Component impedances in s-domain

Component	Impedance in s-domain
Resistor	R
Capacitor	$1/sC$
Inductor	sL

By substituting impedances with previous expressions, it yields to desired transfer function.

3 EQUALIZER TYPES

When choosing a particular equalizer, there are some points that are mainly interesting when deciding what type to use. I would like to reveal some properties of typical equalizer circuits. Drawback of every equalizer is that it requires more powerful amplifier and of course, more excursion capable driver.

3.1 Low-Q equalizer

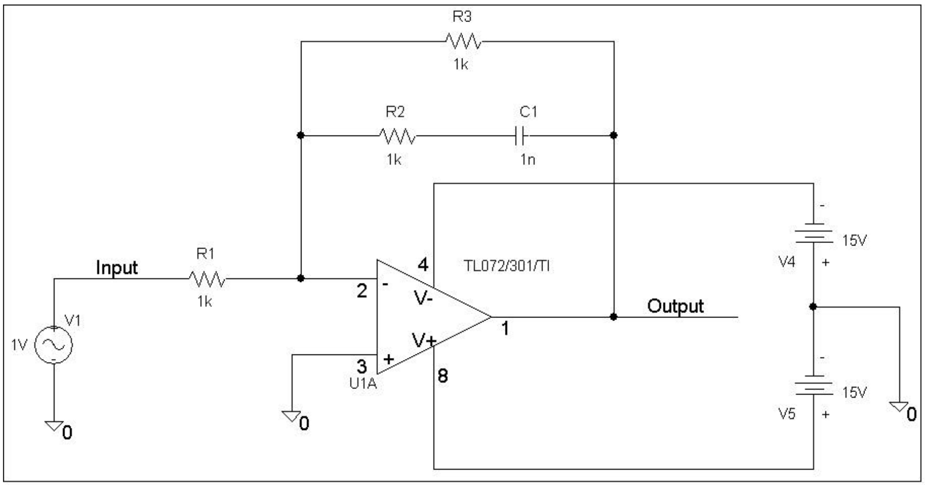

Figure 4. Schematic of modified Linkwitz-equalizer.

3.1.1 Circuit analysis

Circuit is very similar to inverting op-amp circuit, so it can be analyzed by reducing impedances to following basic form:

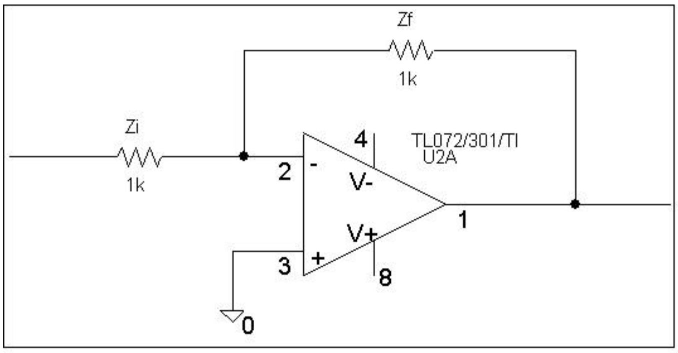

Figure 5. Generalization of inverting opamp circuit impedances.

In figure 5, the transfer function has the following form:

$$G(s) = -\frac{Z_f(s)}{Z_i(s)} \tag{3.1}$$

By comparing figures 4 and 5 we can obtain following values for Z_f and Z_i $(s = jw)$:

$$Z_i = R_1 \tag{3.2}$$

$$Z_f = \left(R_2 + \frac{1}{sC_1}\right) \| R_3 \tag{3.3}$$

By substituting eqs (3.2) and (3.3) to (3.1) we obtain the transfer function of the equalizer:

$$G(s) = -\frac{\left(R_2 + \frac{1}{sC_1}\right) \| R_3}{R_1} = -\frac{R_3}{R_1} \cdot \frac{sR_2C_1+1}{s(R_2+R_3)C_1+1} \tag{3.4}$$

During further analysis/synthesis, I'll ignore the minus in front of transfer function, because it only means that circuit inverts phase by 180°. That can be easily arranged in real life by adding an inverting buffer in front of the equalizer.

From the transfer function (3.4), it can be seen that circuit has one real zero and one real pole.

Natural frequency of zero is

$$f_z = \frac{1}{2pR_2C_1} \tag{3.5}$$

Correspondingly, the natural frequency of the pole is

$$f_p = \frac{1}{2p(R_2+R_3)C_1} \tag{3.6}$$

It can be seen from (3.6) that $f_p < f_z$ for all circuit component values. Therefore, this circuit can't be used to increase natural frequencies of the pole, but it does not make sense anyway.

From (3.4) we can derive the gain that circuit approaches asymptotically, when frequency increases towards infinity.

$$G(\infty) = -\frac{R_2 \| R_3}{R_1} = -\frac{R_2R_3}{R_1(R_2+R_3)} \tag{3.7}$$

By denoting

$$\begin{cases} t_1 = R_2 C_1 \\ t_2 = (R_2 + R_3) C_1 \\ K = \dfrac{R_3}{R_1} \end{cases} \tag{3.8}$$

The magnitude of transfer function (3.4) is

$$|H(s)| = \sqrt{H(s)H(s)^*} = \sqrt{K \frac{st_1 + 1}{st_2 + 1} \cdot K \frac{-st_1 + 1}{-st_2 + 1}} = K \sqrt{\frac{(wt_1)^2 + 1}{(wt_2)^2 + 1}} \tag{3.9}$$

Generally, this equalizer has very gentle slopes, therefore delay distortion on signal is minimized. Following graph shows this property:

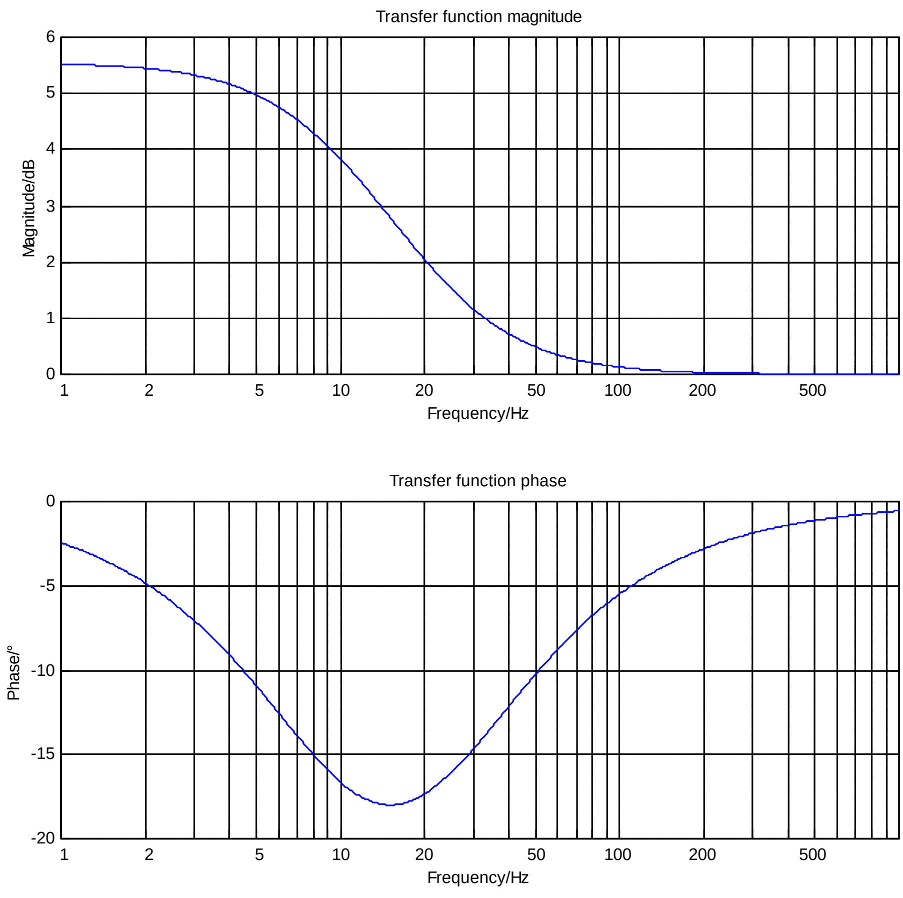

Figure 6. Bode plot of equalizer transfer function.

Because slope of the phase is positive for frequencies greater than about 15 Hz, circuit has negative group delay with those frequencies. Group velocity is therefore superluminal, i.e. greater than c, the speed of light in vacuum (this does not contradict theory of relativity):

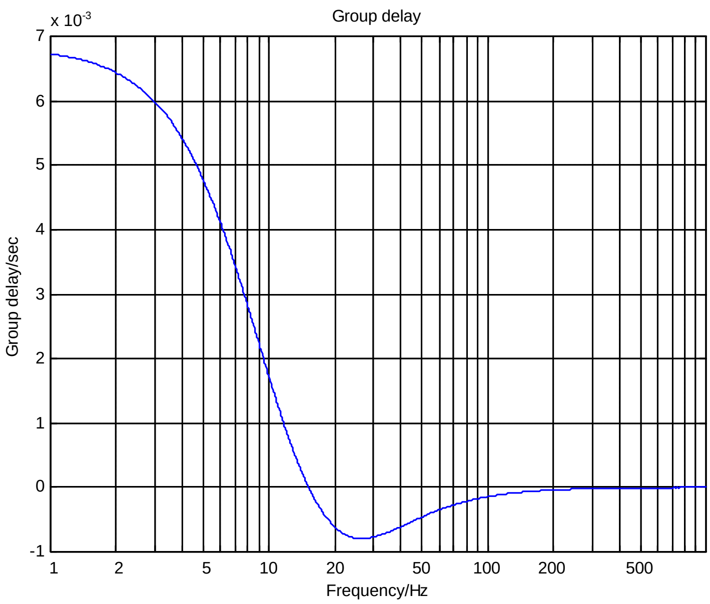

Figure 7. Group delay of equalizer example.

3.1.2 Circuit synthesis

First, determine the poles of existing system. Q_{tc} must be less or equal than 0,5. Determine, if only one or both poles require compensation. As an rule of thumb, a real pole attenuates 6 dB at it's natural frequency. You can calculate attenuation of single pole by using (2.13). Part references refer to figure 4.

To compensate pole use the following procedure:

1) Choose C_1.

A good guess for this capacitor is 100 nF. If resistors become too large (R>>100k), then change capacitor and try again.

2) Calculate R_2 to set natural frequency of the zero introduced by the equalizer.

This frequency should be the same than natural frequency of the pole to be compensated. By solving (3.5) for R_2 we get

$$R_2 = \frac{1}{2pf_z C_1} \tag{3.10}$$

3) Calculate R_3 to set natural frequency of the pole introduced by the equalizer.

This is natural frequency of new pole. Note that it should be lower than frequency of the zero. Again, solving (3.6) for R_3 gives

$$R_3 = \frac{1}{2pf_p C_1} - R_2 \tag{3.11}$$

4) Set the desired high frequency gain by setting R_1.

This can be specified freely, although common practice is to set it to 1 (0 dB gain). Solving (3.7) for R_1 gives:

$$R_1 = \frac{R_2 R_3}{G(\infty)(R_2 + R_3)} \tag{3.12}$$

For more practical design flow, see the example in chapter 4.1.

3.2 High-Q equalizer (”The Linkwitz transform”)

When total system has Q_{tc}>0,5 then system has complex poles, and we can’t use preceding circuit. Fortunately, Linkwitz has designed another form of equalizer [2], which creates one pair of complex zeros and poles. Basically, it compensates poles of the closed box with zeros, and then creates new pair of poles, which are spec’ed by the designer.

Schematic of this equalizer is presented below:

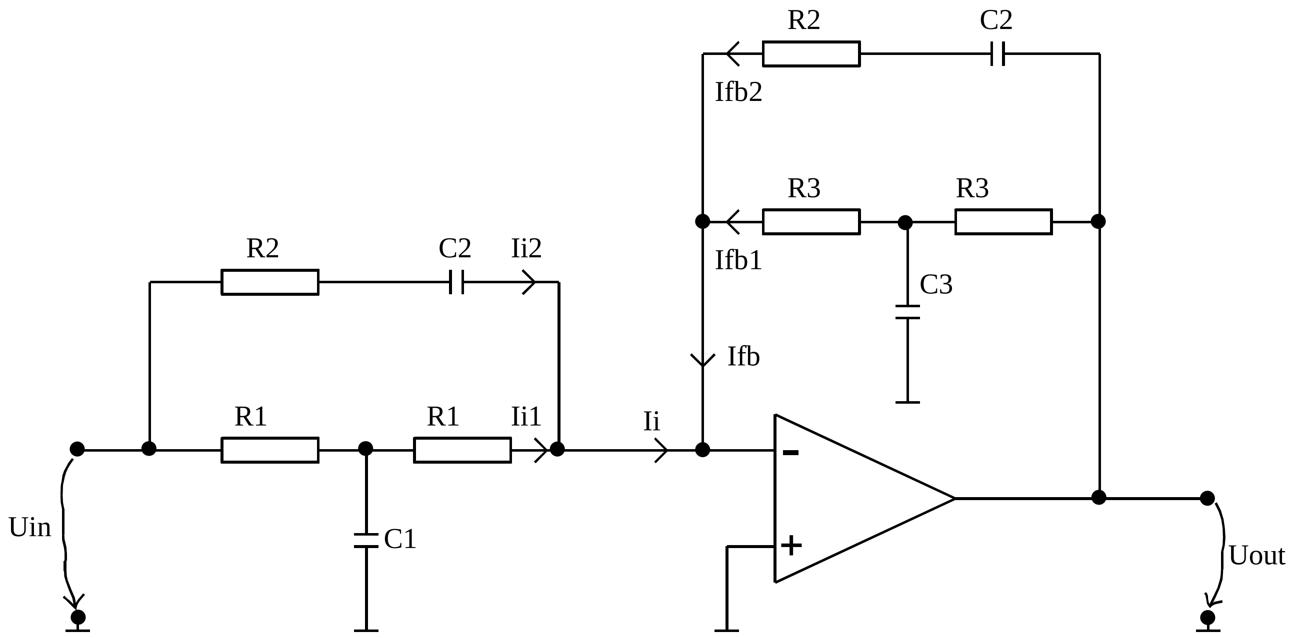

Figure 8. "The Linkwitz Transform", Biquad–type equalizer

3.2.1 Circuit analysis

To derive the transfer function for this type of equalizer, we can use preceding simplification to generalized impedance in feedback loop. This circuit is a bit more difficult, because it has voltage dividers in feedback, so this divider has to be considered separately.

Circuit notation is same that Linkwitz used.

Current I_{i1} is a bit tricky, because it includes filter.

Let's define a temporary variable, $U_c(s)$, for voltage across capacitor C_1. For this voltage we get a following expression, by using the voltage divider equation:

$$U_c(s)=U_{in}(s)\frac{\left(\frac{1}{sC_1}\|R_1\right)}{\left(\frac{1}{sC_1}\|R_1\right)+R_1}=U_{in}(s)\frac{R_1}{R_1^{\ 2}C_1s+2R_1} \tag{3.8}$$

Now, the current I_{i1} is simply

$$I_{i_1}(s)=\frac{U_c(s)}{R_1}=U_{in}(s)\frac{R_1}{R_1^{\ 2}C_1s+2R_1}\frac{1}{R1}=U_{in}(s)\frac{1}{R_1^{\ 2}C_1s+2R_1} \tag{3.9}$$

For input current I_{i2}, we get following expressions:

$$I_{i_2}(s)=U_{in}(s)\frac{1}{Z}=U_{in}(s)\frac{1}{R_2+\frac{1}{sC_2}}=U_{in}(s)\frac{C_2s}{C_2R_2s+1} \tag{3.10}$$

Combining these currents gives us the total input current I_i:

$$\begin{aligned}I_i(s)&=I_{i_1}(s)+I_{i_2}(s)=U_{in}(s)\left(\frac{1}{R_1C_1s+2R_1}+\frac{C_2s}{C_2R_2s+1}\right)\\&=U_{in}(s)\frac{C_1C_2R_1^{\ 2}s^2+(2C_2R_1+C_2R_2)s+1}{\mathrm{C_1C_2R_1^{\ 2}R_2s^2+\left(C_1R_1^{\ 2}+2C_2R_1R_2\right)s+2R_1}}\end{aligned} \tag{3.11}$$

Because feedback network is similar, we obtain for I_{fb} using same techniques

$$\begin{aligned}I_{fb}(s)&=I_{fb_1}(s)+I_{fb_2}(s)=U_{out}(s)\left(\frac{1}{R_3C_3s+2R_1}+\frac{C_2s}{C_2R_2s+1}\right)\\&=U_{out}(s)\frac{C_2C_3R_3^{\ 2}s^2+(2C_2R_3+C_2R_2)s+1}{\mathrm{C_3C_2R_3^{\ 2}R_2s^2+\left(C_3R_3^{\ 2}+2C_2R_3R_2\right)s+2R_3}}\end{aligned} \tag{3.12}$$

Now, because we can assume that inverting input of the opamp doesn't take any current from that node, we can combine (3.11) and (3.12) by using Kirchoff's current law:

$$I_i(s)+I_{fb}(s)=0 \tag{3.13}$$

Substituting (3.11) and (3.12) into (3.13) yields to

$$\begin{aligned}&U_{in}(s)\frac{C_1C_2R_1^{\ 2}s^2+(2C_2R_1+C_2R_2)s+1}{\mathrm{C_1C_2R_1^{\ 2}R_2s^2+\left(C_1R_1^{\ 2}+2C_2R_1R_2\right)s+2R_1}}+\\&U_{out}(s)\frac{C_2C_3R_3^{\ 2}s^2+(2C_2R_3+C_2R_2)s+1}{\mathrm{C_3C_2R_3^{\ 2}R_2s^2+\left(C_3R_3^{\ 2}+2C_2R_3R_2\right)s+2R_3}}=0\end{aligned} \tag{3.14}$$

Solving transfer function from (3.14) gives

$$H(s) = -\frac{R_3}{R_1}\frac{(C_3R_3s+2)}{(C_1R_1s+2)}\frac{\left(C_1C_2R_1{}^2s^2+(2C_2R_1+C_2R_2)s+1\right)}{\left(C_2C_3R_3{}^2s^2+(2C_2R_3+C_2R_2)s+1\right)} \tag{3.15}$$

From (3.15) it is seen, that circuit has one real pole and zero, and one second order pole and zero, which could be complex or real. When designing this type of equalizer, it is important that natural frequencies of the real pole and zero are the same, because otherwise they introduce some warping to frequency response.

Real zero is therefore

$$C_3R_3s+2=0 \Leftrightarrow s_{z_1} = -\frac{2}{C_3R_3} \tag{3.16}$$

and real pole is

$$C_1R_1s+2=0 \Leftrightarrow s_{p_1} = -\frac{2}{C_1R_1} \tag{3.17}$$

For the second order factors, we can use (2.4) and (2.6) to determine f_0, Q_0, f_p and Q_p.

Natural frequency of real zero by using (2.15) is

$$f_{z_1} = \frac{|s_{z1}|}{2p} = \frac{\frac{2}{C_3R_3}}{2p} = \frac{1}{pC_3R_3} \tag{3.18}$$

and natural frequency of real pole is

$$f_{p_1} = \frac{|s_{p1}|}{2p} = \frac{\frac{2}{C_1R_1}}{2p} = \frac{1}{pC_1R_1} \tag{3.19}$$

for second order zero, the coefficients $a_0 \ldots a_2$ are:

$$\begin{cases} a_0 = 1 \\ a_1 = 2C_2R_1 + C_2R_2 \\ a_2 = C_1C_2R_1{}^2 \end{cases} \tag{3.20}$$

By using (2.4) and (2.6) we get f_0 and Q_0 from second order factor of numerator of (3.15):

$$f_0 = \frac{w_0}{2p} = \frac{\sqrt{\frac{a_0}{a_2}}}{2p} = \frac{\sqrt{\frac{1}{C_1C_2R_1{}^2}}}{2p} = \frac{1}{2pR_1\sqrt{C_1C_2}} \tag{3.21}$$

$$Q_0 = \frac{\sqrt{a_0 a_2}}{a_1} = \frac{\sqrt{1 \cdot C_1 C_2 R_1^{\ 2}}}{2C_2 R_1 + C_2 R_2} = \frac{R_1}{2R_1 + R_2} \cdot \frac{\sqrt{C_1 C_2}}{C_2} = \frac{R_1}{2R_1 + R_2} \sqrt{\frac{C_1}{C_2}} \tag{3.22}$$

Similarly, f_p and Q_p is determined from second order factor in denominator of (3.15):

$$\begin{cases} a_0 = 1 \\ a_1 = 2C_2 R_3 + C_2 R_2 \\ a_2 = C_2 C_3 R_3^{\ 2} \end{cases} \tag{3.23}$$

$$f_p = \frac{w_p}{2p} = \frac{\sqrt{\frac{a_0}{a_2}}}{2p} = \frac{\sqrt{\frac{1}{C_2 C_3 R_3^{\ 2}}}}{2p} = \frac{1}{2pR_3\sqrt{C_2 C_3}} \tag{3.24}$$

$$Q_p = \frac{\sqrt{a_0 a_2}}{a_1} = \frac{\sqrt{1 \cdot C_2 C_3 R_3^{\ 2}}}{2C_2 R_3 + C_2 R_2} = \frac{R_3}{2R_3 + R_2} \cdot \frac{\sqrt{C_2 C_3}}{C_2} = \frac{R_3}{2R_3 + R_2} \sqrt{\frac{C_3}{C_2}} \tag{3.25}$$

This equalizer does not make delay distortion worse. Bode plot of transfer function is shown below. F_0 is 70 Hz, Q_0 is 0.9, f_p is 18 Hz and Q_p 0.707:

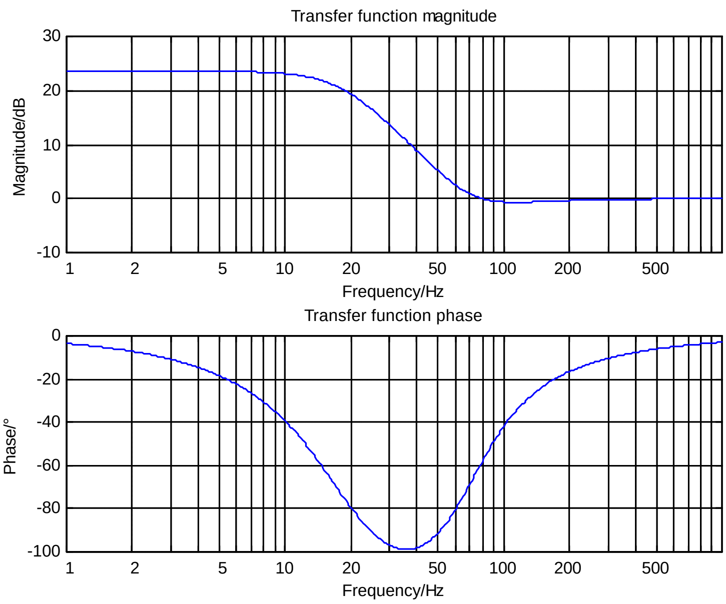

Figure 9. Transfer function bode plot for Linkwitz transform equalizer.

This equalizer also exhibits a negative group delay behaviour:

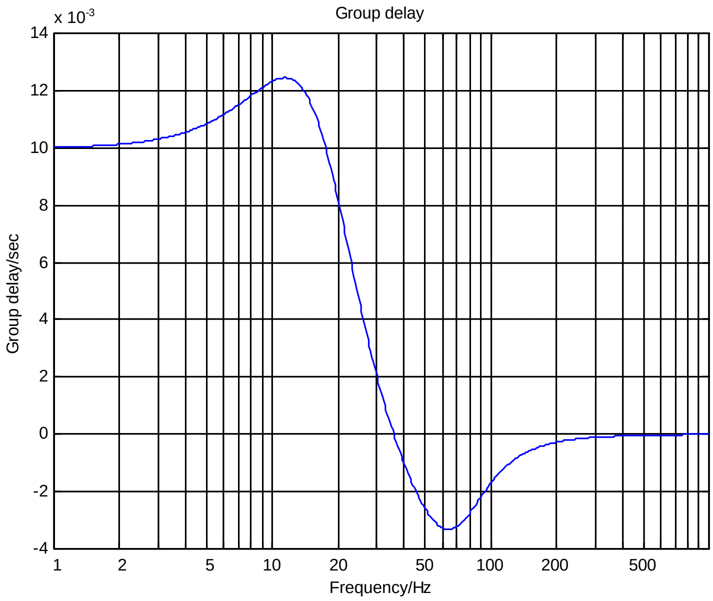

Figure 10. Group delay behaviour of linkwitz transform equalizer.

3.2.2 Circuit synthesis

The design procedure to this equalizer is presented by Linkwitz [1]. I'll include it here for completeness.

1) Specify f_0,Q_0, f_p and Q_p.

f_0 and Q_0 is determined from closed box design. They spec the exact compensation for existing poles. These values are often given from any decent speaker simulation software, WinISD for example, there Q_{tc} is the value from Q_0 and f_c is the value for f_0, respectively.

Specify reasonable valued for Q_p and f_p.

2) Calculate constant 'k'

It is required that k is positive, for a realizable equalization using this circuit topology.

$$k = \frac{\frac{f_0}{f_p} - \frac{Q_0}{Q_p}}{\frac{Q_0}{Q_p} - \frac{f_p}{f_0}} \tag{3.26}$$

3) Choose C_2.

4) Calculate R_1.

$$R_1 = \frac{1}{2pf_0 C_2 \left(2Q_0 \left(1+k\right)\right)} \tag{3.27}$$

5) Calculate R_2.

$$R_2 = 2kR_1 \tag{3.28}$$

6) Calculate C_1.

$$C_1 = C_2\left(2Q_0(1+k)\right)^2 \tag{3.29}$$

7) Calculate C_3.

$$C_3 = C_1\left(\frac{f_p}{f_0}\right)^2 \tag{3.30}$$

8) Calculate R_3.

$$R_3 = R_1\left(\frac{f_0}{f_p}\right)^2 \tag{3.31}$$

If the resistors become too large, i.e. >>100k, then change the capacitor value and try again.

3.3 ”High-Q_p” second order high-pass filter equalizer

This type of equalizer is often used for ported boxes, because it also works as subsonic filter. It has some undesirable features which I’ll explain in following chapter. For finnish readers, this type of equalizer is used in HIFI 100/1 active subwoofer crossover [3].

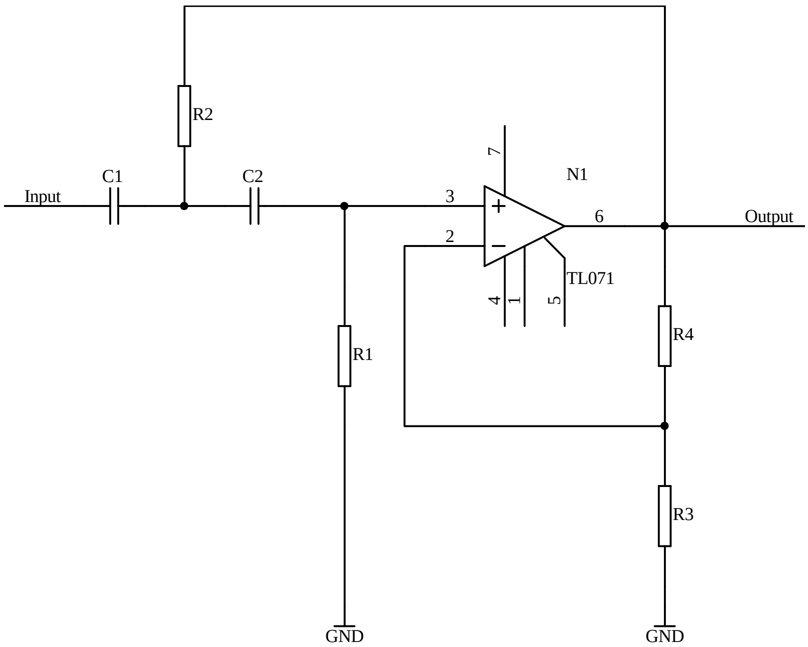

Figure 11. Schematic of high-pass filter equalizer

3.3.1 Circuit analysis

By using part references in figure 7, the transfer function is ([2] p.9):

$$H(s) = K\frac{R_1R_2C_1C_2s^2}{R_1R_2C_1C_2s^2 + (R_2C_2 + R_2C_1 + R_1C_2(1-K))s + 1} \tag{3.32}$$

with

$$K = \frac{R_3 + R_4}{R_3} \tag{3.33}$$

Denominator coefficients are therefore

$$\begin{cases} a_0 = 1 \\ a_1 = R_2C_2 + R_2C_1 + R_1C_2(1-K) \\ a_2 = R_1R_2C_1C_2 \end{cases} \tag{3.34}$$

By using (2.4) and (2.6) we get f_n and Q:

$$f_n = \frac{w_n}{2p} = \frac{\sqrt{\frac{a_0}{a_2}}}{2p} = \frac{\sqrt{\frac{1}{R_1R_2C_1C_2}}}{2p} = \frac{1}{2p\sqrt{R_1R_2C_1C_2}} \tag{3.35}$$

$$Q = \frac{\sqrt{a_0a_2}}{a_1} = \frac{\sqrt{1\cdot R_1R_2C_1C_2}}{R_2C_2 + R_2C_1 + R_1C_2(1-K)} = \frac{\sqrt{R_1R_2C_1C_2}}{R_2C_2 + R_2C_1 + R_1C_2(1-K)} \tag{3.36}$$

In HIFI 100/1 crossover equalizer [3], stage that performs actual response correction has variable gain, and C_1=C, C_2=C, R_1=R, R_2=R. With these substitutions, (3.35) and (3.36) can be simplified as follows:

$$f_n = \frac{1}{2p\sqrt{RRCC}} = \frac{1}{2pRC} \tag{3.37}$$

$$Q = \frac{\sqrt{RRCC}}{RC + RC + RC(1-K)} = \frac{RC}{3RC - KRC} = \frac{RC}{RC(3-K)} = \frac{1}{3-K} \tag{3.38}$$

Let’s analyze its time domain properties via step and tone burst responses. As it was designed and published in 1988, it has following values for R and C:

R=9,1 kΩ; C=1 µF.

With these values, natural frequency for that equalizer is set to

$$f_n = \frac{1}{2pRC} = \frac{1}{2p \cdot 9{,}1\mathrm{kO} \cdot 1\mu\mathrm{F}} = 17{,}5Hz \tag{3.39}$$

The gain is arranged to be variable so that, R_3 is fixed to 3,9 kΩ. R_4 is 560 + 10k potentiometer and 15 k fixed resistor in parallel. Therefore, R_4 can be varied from 560 to

$$R_{4\max} = 560\,\Omega + \frac{1}{\frac{1}{10\,\mathrm{k}\Omega} + \frac{1}{15\,\mathrm{k}\Omega}} = 560\Omega + 6\mathrm{k}\Omega = 6{,}56\mathrm{k}\Omega \quad (3.40)$$

Gain is therefore variable between

$$K_{\min} = \frac{R_3 + R_{4\min}}{R_3} = \frac{3{,}9\mathrm{k}\Omega + 560\Omega}{3{,}9\mathrm{k}\Omega} = 1{,}1436 \quad (3.41)$$

and

$$K_{\max} = \frac{R_3 + R_{4\max}}{R_3} = \frac{3{,}9\mathrm{k}\Omega + 6{,}560\mathrm{k}\Omega}{3{,}9\mathrm{k}\Omega} = 2{,}6821 \quad (3.42)$$

And Q is variable from

$$Q_{\min} = \frac{1}{3 - K_{\min}} = \frac{1}{3 - 1{,}1436} = 0{,}5387 \quad (3.43)$$

to

$$Q_{\max} = \frac{1}{3 - K_{\max}} = \frac{1}{3 - 2{,}6821} = 3{,}1456 \quad (3.44)$$

It has also another second order highpass filter. In this stage, R_3 is also 3,9 kΩ and R_4 is 4,7 kΩ. So gain is

$$K = \frac{R_3 + R_4}{R_3} = \frac{3{,}9\mathrm{k}\Omega + 4{,}7\mathrm{k}\Omega}{3{,}9\mathrm{k}\Omega} = 2{,}2051 \quad (3.45)$$

so Q is

$$Q = \frac{1}{3 - K} = \frac{1}{3 - 2{,}2051} = 1{,}2581 \quad (3.46)$$

Frequency response for these two cases (Q in min and max) is presented in figure 12. Please note that this graph includes only the equalizer and high pass filter stage responses, so actual response is a bit different.

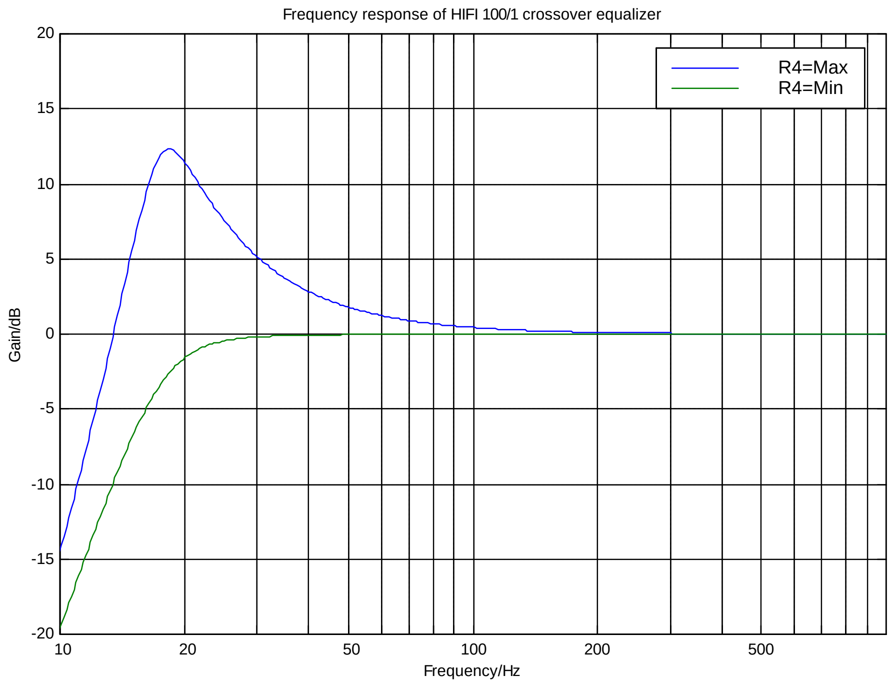

Figure 12. Frequency response of HIFI 100/1 crossover equalizer.

Because original design ported box was tuned to 25 Hz, this design overloads driver easily, because it's maximum gain is some 17 Hz.

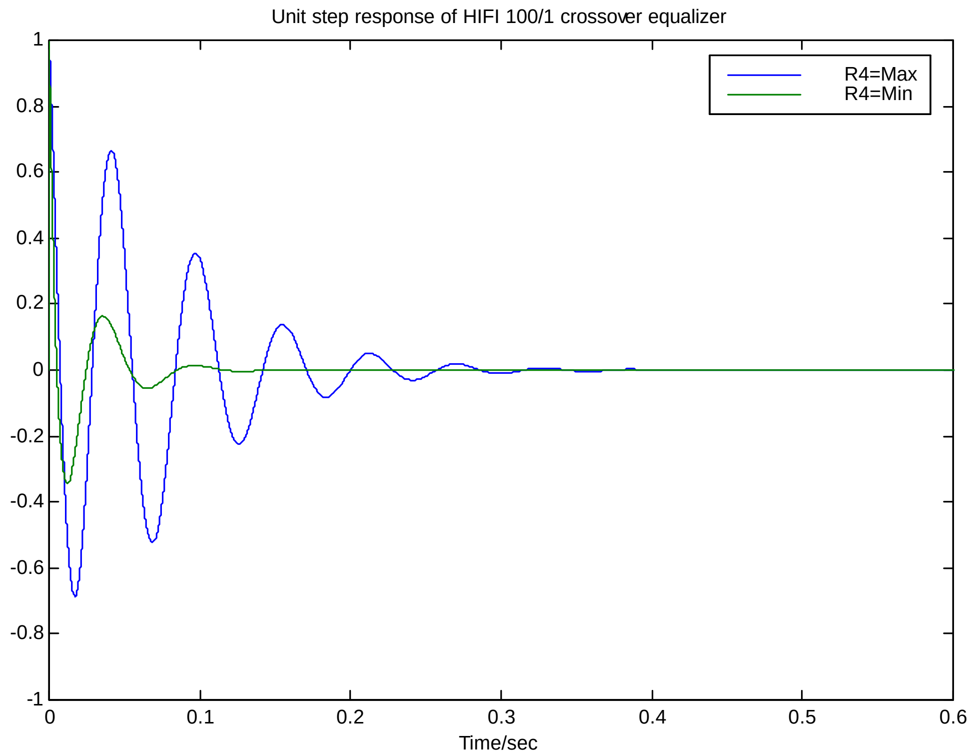

Figure 13. Unit step response of HIFI 100/1 crossover equalizer.

From figure 13, it is evident why this type of equalizer is not particulary good in terms of transient response. Group delay graph supports this:

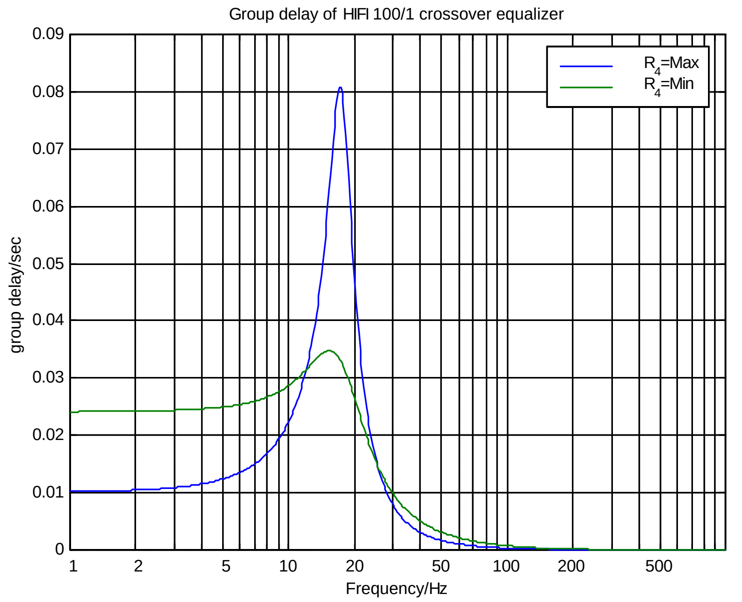

Figure 14. Group delay of HIFI 100/1 crossover equalizer.

For burst testing, I used five-cycle cosine shaped tone burst signal. Frequency of underlying sine signal is 20 Hz.

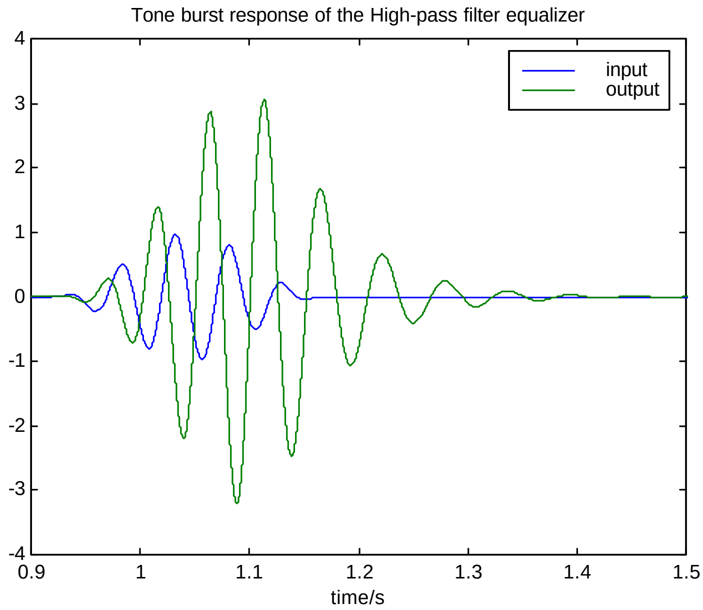

Figure 15. Tone burst response of high-pass filter equalizer.

It is clear that this type of equalizer has quite high amount of ringing.

3.3.2 Circuit synthesis

When designing such an equalizer, one should first decide how circuit should be configured. Basically there are two commonly used options ([2] mentions 2 more):

1) Set filter components as equal, as in previous example of HIFI 100/1.

With this option, gain of the opamp is normally set to 1, so in high frequencies circuit presents no change to signal. Now, choose C and f_n and calculate R from

$$R = \frac{1}{2pf_n C} \tag{3.42}$$

Choose equalizer Q and calculate required gain with

$$K = 3 - \frac{1}{Q} \tag{3.43}$$

Then choose either resistor (R_3 or R_4) and calculate another from

$$R_3 = \frac{R_4}{K-1} \Leftrightarrow R_4 = R_3(K-1) \tag{3.44}$$

2) Set Resistors as ratios and capacitors equal.

3.4 Integrator equalizer (ELF)

A closed box loudspeaker is well below its resonance, a dual differentiator. The differentiator’s transfer function is:

$$H(s) = s \tag{3.45}$$

This single s produces zero at the origin of the s-plane. So there are double zero at origin of any closed box speaker. This determines the rising slope of any loudspeaker response.

The idea behind ELF is that, when two zeros at the origin are eliminated by adding double integrator to the system, then the design is converted to equivalent low-pass filter. Schematics for ELF double integrator is shown in figure 16.

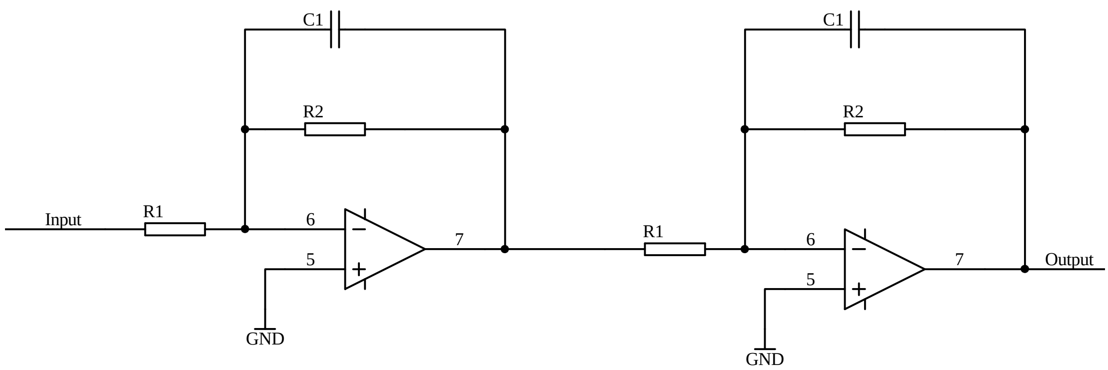

Figure 16. ELF double integrator schematics.

Transfer function for single integrator is

$$H(s)=\frac{1}{s} \tag{3.46}$$

Integrator therefore introduces a pole in origin, which cancels zero. Circuit in figure approximates square of (3.45). With what differences, we'll soon find out. Interesting property of (3.46) is that circuit introduces no delay distortion on signal passing by. This is because phase shift is constant (group delay is defined as $\mathrm{gd}(w)=-\frac{df}{dw}$, remember?).

3.4.1 Circuit analysis

To derive the transfer function for ELF, we can (again) use impedance generalization described in figure 5. Impedances for (3.1) therefore are (per opamp section):

$$\begin{cases} Z_f = \dfrac{1}{\dfrac{1}{R_2}+sC_1} \\ Z_i = R_1 \end{cases} \tag{3.47}$$

So transfer function is (per section):

$$H(s)=-\frac{Z_f}{Z_i}=-\frac{\dfrac{1}{\dfrac{1}{R_2}+sC_1}}{R_1}=-\frac{\dfrac{\dfrac{1}{1+sR_2C_1}}{R_2}}{R_1}=-\frac{R_2}{R_1}\cdot\frac{1}{R_2C_1s+1} \tag{3.48}$$

By substituting R_2C_1 as τ, then we get

$$H(s)=-\frac{R_2}{R_1}\cdot\frac{1}{ts+1}=-\frac{R_2}{R_1}\cdot\frac{1}{s+\dfrac{1}{t}}=-\frac{R_2}{R_1}\cdot\frac{1}{s+w_n} \tag{3.49}$$

where ω_n is defined as

$$w_n=\frac{1}{R_2C_1} \tag{3.50}$$

Now, it can be seen that when natural frequency ω_n is close to zero, (3.48) is a good approximation to (3.46).

3.4.2 Circuit synthesis

3.5 Low-pass filtering equalizer

Low pass filter resembles somewhat to ELF equalizer. Idea is to put corner frequency low enough, so it practically is almost like an integrator.

3.5.1 Circuit analysis

3.5.2 Circuit synthesis

4 SOME EXAMPLES

4.1 Equalization of the low-Q ($Q_{tc} \leq 0.5$) design

I have chosen Peerless XLS-10 (830457) driver for this example, mainly because I think it serves also as building instructions for such an equalizer. I have not seen very many closed box designs with this driver, that is probably because it gives need to equalize it and because it is not very commonly practised art, designs won't simply exist. Albeit from this difficulty, closed box gives you smallest time domain distortion on waveform (best transient response, to put it more simply). I have included Thiele-Small parameters for the XLS-10 driver for convenience in table 2.

Table 2. Parameters for Peerless XLS-10 driver

Parameter	Value
Q_{es}	0,18
Q_{ms}	2,63
Q_{ts}	0,17
V_{as}	89,7 l
F_s	18,9 Hz
Bl	17,5 Tm
S_d	352 cm^2
R_e	3.4 Ω
L_e	4.3 mH
X_{max}	12.5 mm
P_e	350 W

For the box volume, I chose 35 litres, because it is not so small, that is difficult to construct and it is also quite small for even small rooms.

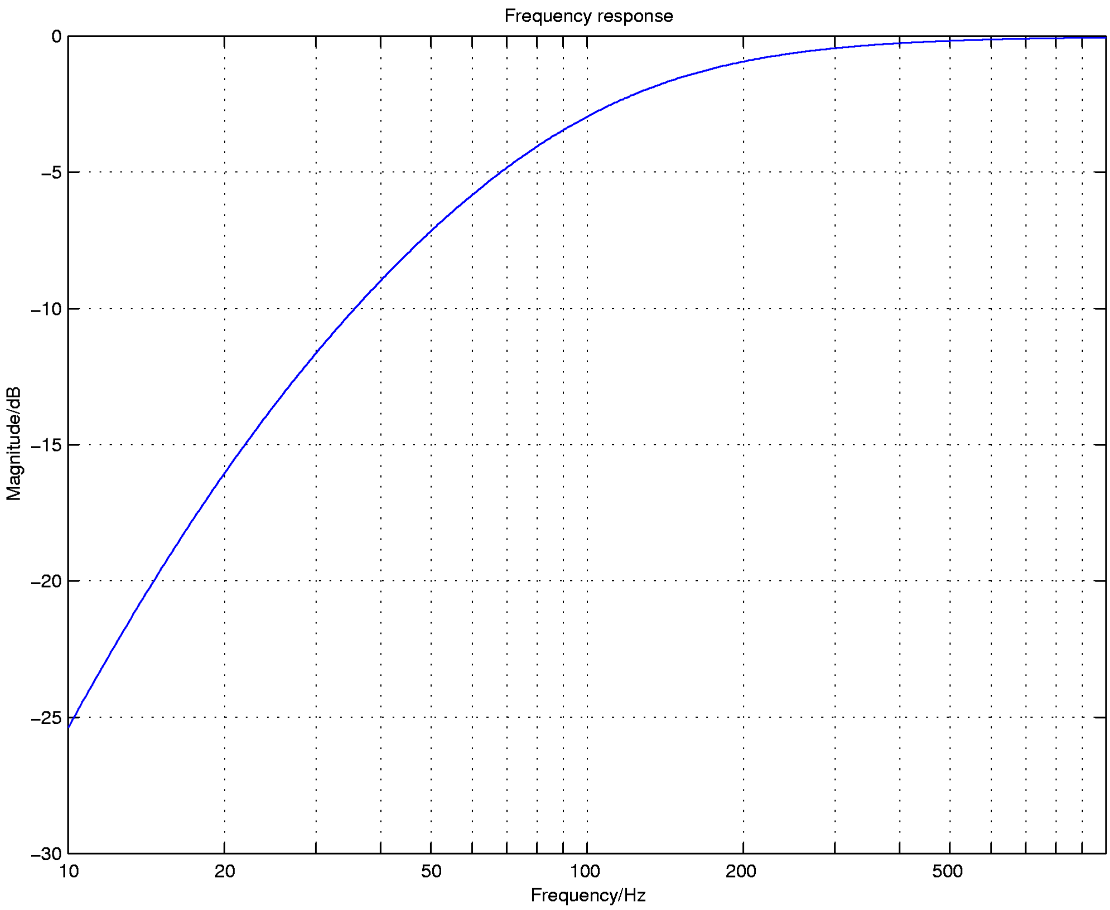

Figure 17. Free-field frequency response without any equalization.

The system has a transfer function (see my another paper about how to derive this):

$$H(s)=\frac{1.751\cdot 10^{-11}s^2}{1.751\cdot 10^{-11}s^2+1.234\cdot 10^{-8}s+8.796\cdot 10^{-7}} \tag{4.1}$$

so transfer function coefficients for (2.2) are:

$$\begin{cases} a_2=1.751\cdot 10^{-11} \\ a_1=1.234\cdot 10^{-8} \\ a_0=8.796\cdot 10^{-7} \end{cases} \tag{4.2}$$

Solving denominator polynomial roots by using standard quadratic equation solving formula gives us the system poles:

$$\begin{cases} p_1=\dfrac{-a_1+\sqrt{a_1^{\,2}-4a_2a_0}}{2a_2}=\dfrac{-1.234\cdot 10^{-8}+\sqrt{\left(1.234\cdot 10^{-8}\right)^2-4\cdot 1.751\cdot 10^{-8}\cdot 8.796\cdot 10^{-7}}}{2\cdot 1.751\cdot 10^{-8}}=\text{-}80.47 \\ p_2=\dfrac{-a_1-\sqrt{a_1^{\,2}-4a_2a_0}}{2a_2}=\dfrac{-1.234\cdot 10^{-8}-\sqrt{\left(1.234\cdot 10^{-8}\right)^2-4\cdot 1.751\cdot 10^{-8}\cdot 8.796\cdot 10^{-7}}}{2\cdot 1.751\cdot 10^{-8}}=-624.2 \end{cases}$$

(4.3)

Pole-zero map is shown in Figure 18.

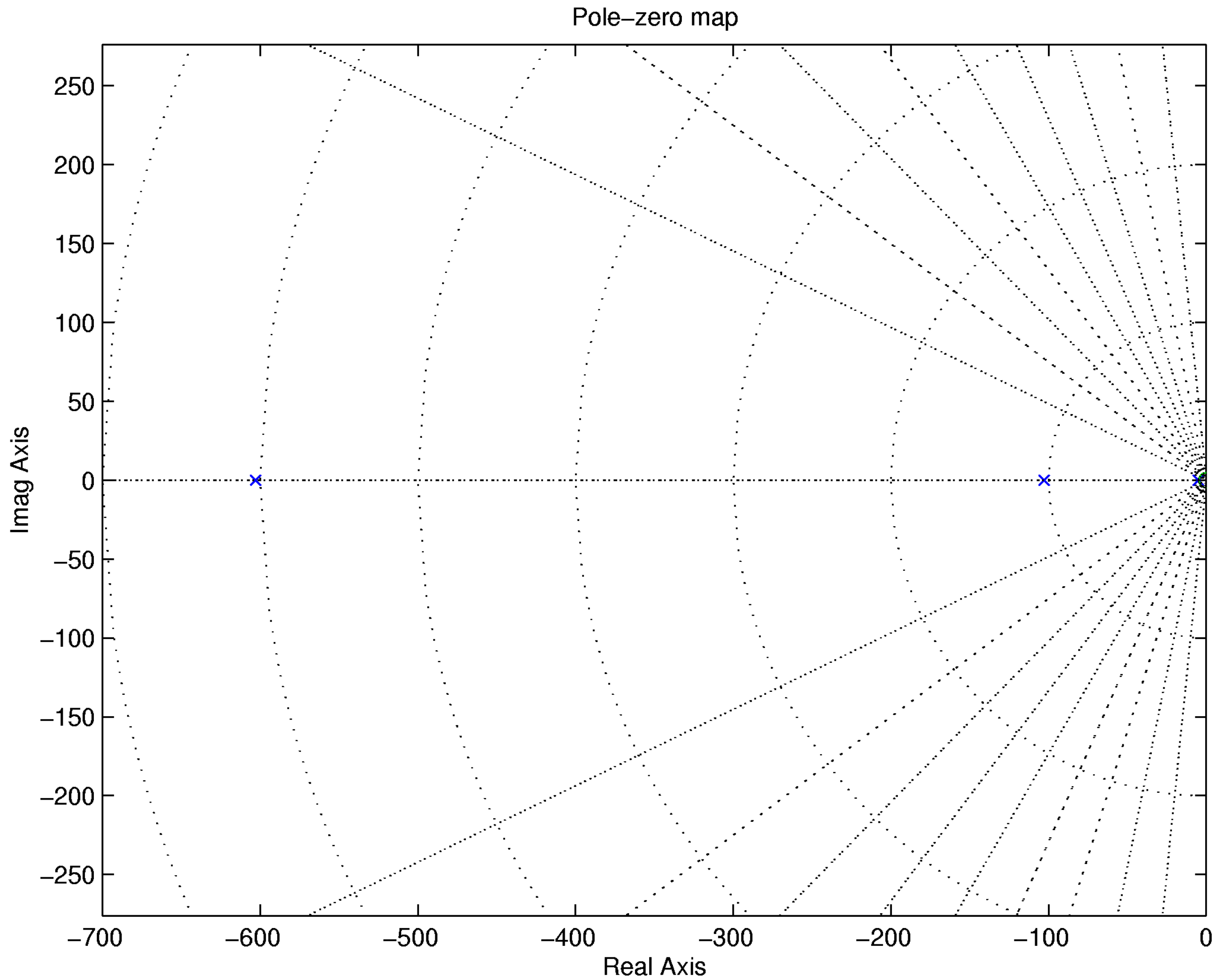

Figure 18. Pole-zero map of the unequalized design.

Solving Q_{tc} and f_{sc} using (2.4) and (2.6) gives

$$f_{sc} = \frac{1}{2p\sqrt{\frac{1}{80.47}\cdot\frac{1}{624.2}}} = 35{,}67\,\mathrm{Hz} \tag{4.4}$$

$$Q_{tc} = \frac{\sqrt{\frac{1}{80.47}\cdot\frac{1}{624.2}}}{\frac{1}{80.47}+\frac{1}{624.2}} = 0{,}318 \tag{4.5}$$

Natural frequencies for these poles are modulus of each pole:

$$w_{p_1} = |p_1| = |-80.47| = 80.47\,\frac{rad}{s} = 12.80\ \text{Hz}$$
$$w_{p2} = |p_2| = |-624.2| = 624.2\,\frac{rad}{s} = 99.35\ \text{Hz} \qquad (4.6)$$

Note that frequencies mentioned below are slightly different because higher accuracy coefficients (pole natural frequencies calculated directly with Matlab).

Because only p_2 has unconfortably high natural frequency, let's add compensation for that. We'll chose frequency near p_1's natural frequency.

To find required frequency, let's equalize this design so that it's -3 dB frequency is 20 Hz. To find attenuation of lower frequency pole at 20 Hz, let's calculate its magnitude at 20 Hz by using (2.13):

$$\left|H_{p_1}(s)\right| = \left|\frac{s}{s-(-80.47)}\right|_{s=j2p\,20} = \frac{(2p\,20)^2}{\sqrt{(-80.47)^2 + (2p\,20)^2}} = 0{,}8421 \qquad (4.7)$$

So, because –3 dB attenuation is about 0,707, we can allow additional attenuation of 0,84066 for second pole.

Solving (2.13) for p, we get

$$p = -\frac{w\sqrt{1-\left|H_{p1}(s)\right|^2}}{\left|H_{p1}(s)\right|} = -\frac{2p\,20\sqrt{1-(0{,}84066)^2}}{0{,}84066} = \text{-80,95} \qquad (4.8)$$

Natural frequency for required new pole is

$$f_{pd} = \frac{-p}{2p} = \frac{-(-80{,}95)}{2p} = 12{,}88\text{Hz} \qquad (4.9)$$

To cancel a pole, we must locate a zero to just a top of a zero. Pole p_2 has highest natural frequency, so it is desirable to compensate that.

Let's take a look into group delay and unit step graphs before going to detailed design of the equalizer.

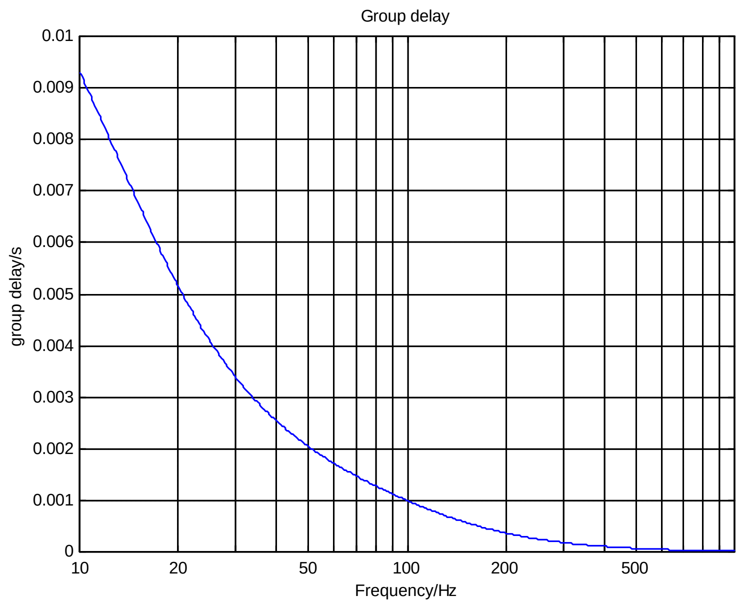

Figure 19. Group delay is very small, even for low frequencies.

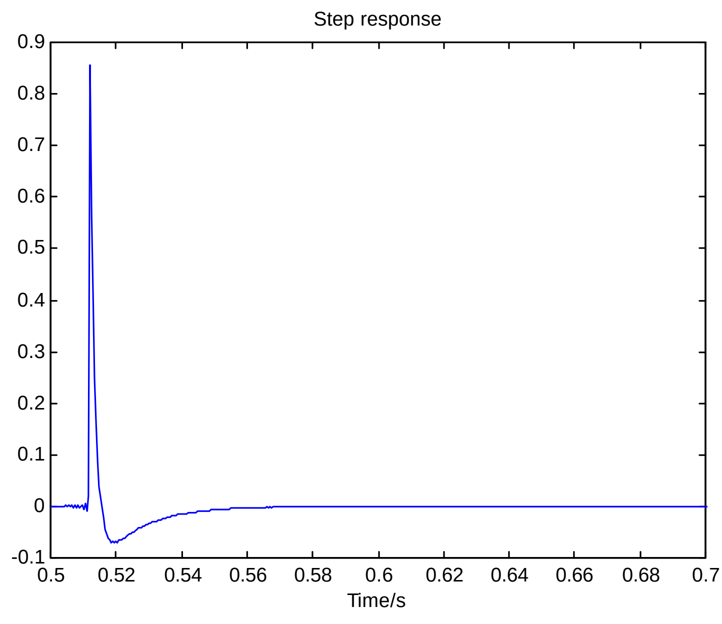

Figure 20. Step response.

Step response shows no overshoot. This is because $Q_{tc}<0,5$.

Let's chose 100 nF capacitor for C_1 by using "stetson-method" (only advanced designers should use it because it is so powerful technique), and by setting the circuit's zero natural frequency to same value as box pole p_2 we can calculate value for R_2:

$$R_2 = \frac{1}{2\pi f_{p2} C_1} = \frac{1}{2\pi \cdot 99{,}38\,Hz \cdot 100\,nF} = \underline{16014{,}79\,\Omega} \tag{4.10}$$

now we can obtain value for R_3 with R_2 and frequency for desired final pole frequency (f_{pd}):

$$R_3 = \frac{1}{2\pi f_{pd} C_1} - R_2 = \frac{1}{2\pi \cdot 12{,}88\,Hz \cdot 100\,nF} - 16014{,}79\,\Omega = \underline{107514{,}35\,\Omega} \tag{4.11}$$

R_1 is chosen so, that gain approaches unity in high frequency range (other gains are possible, just change it):

$$R_1 = \frac{R_2 R_3}{G(\infty)(R_2 + R_3)} = \frac{16014{,}79\,\Omega \cdot 107514{,}35\,\Omega}{1 \cdot (16014{,}79\,\Omega + 107514{,}35\,\Omega)} = \underline{13938{,}57\,\Omega} \tag{4.12}$$

The final schematic for equalizer is following:

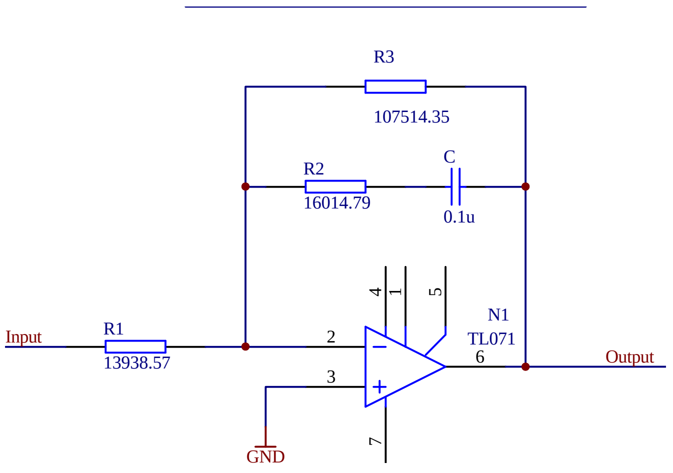

Figure 21. Schematics for equalizer.

R_1 is a bit small for using this circuit directly between subwoofer power amplifier and filter, so it is advisable to precede this stage with inverting buffer stage, which has gain of –1. It also corrects inverting behaviour of this circuit. It is also necessary to choose R_1-R_3 from standard resistor series, such as E96.

Equalizer transfer function magnitude in dB is shown below:

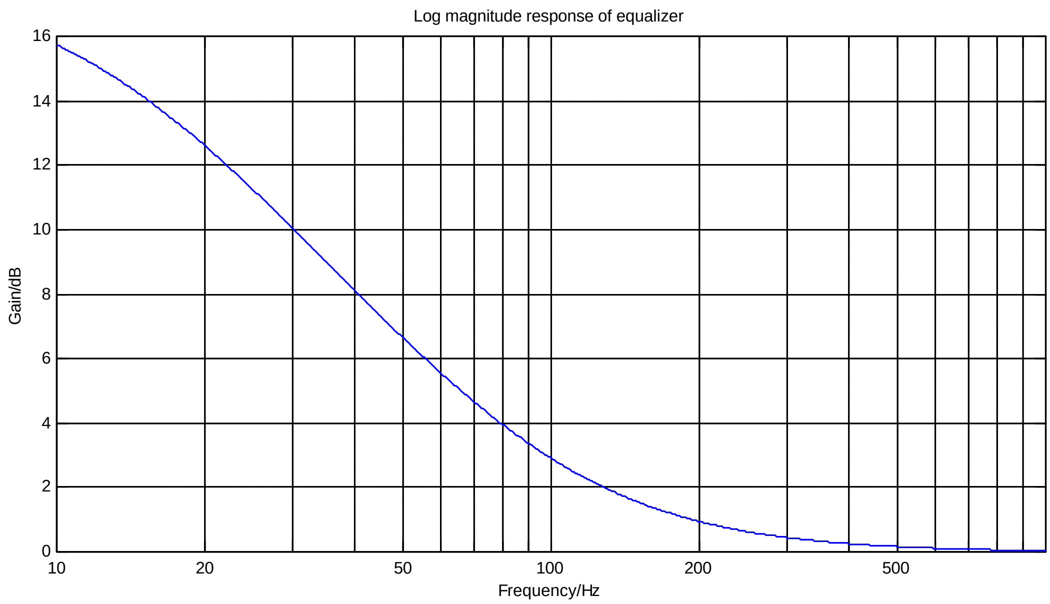

Figure 22. Equalizer transfer function magnitude in dB.

Equalized system transfer function magnitude compared to unequalized system is shown below:

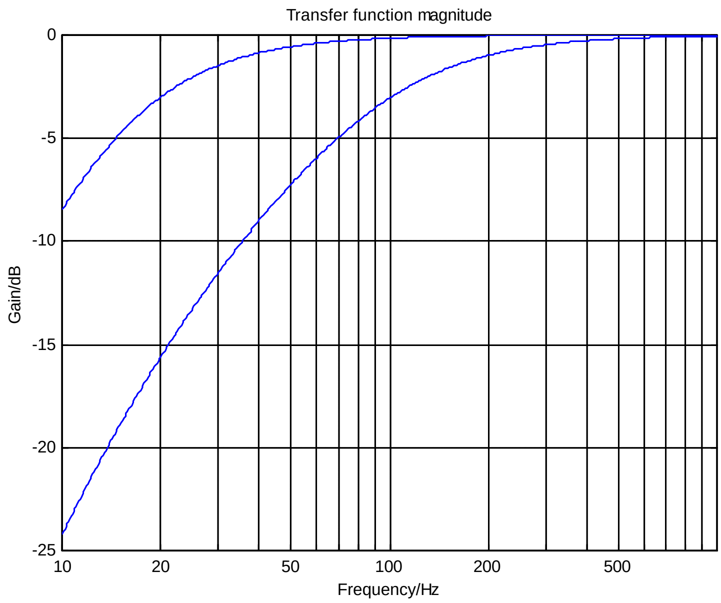

Figure 23. Equalized system versus unequalized. Unequalized system shown dashed.

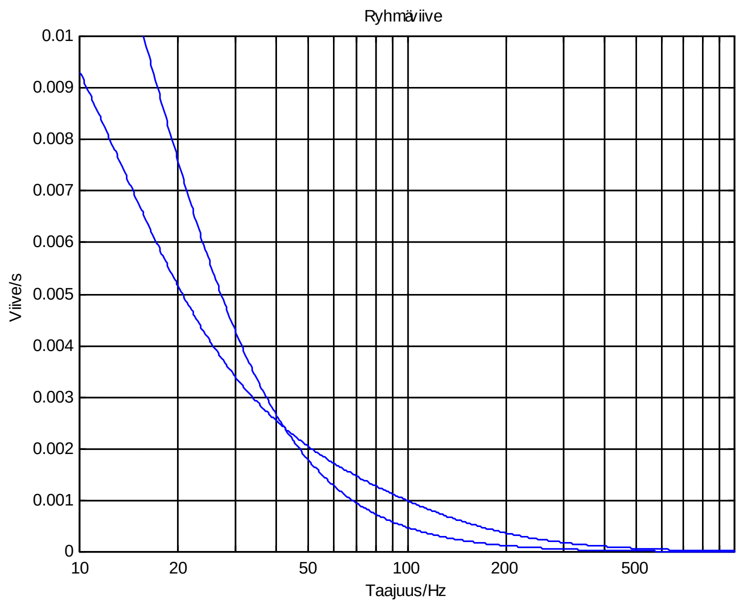

Figure 24. Group delay in unequalized and equalized cases.

From the group delay graph in figure 24 it is shown that group delay increases a bit, but it is still very small. Funnily, group delay even becomes smaller in high frequencies.

Pole-zero diagram below shows how equalizer zero compensates leftmost pole of unequalized system:

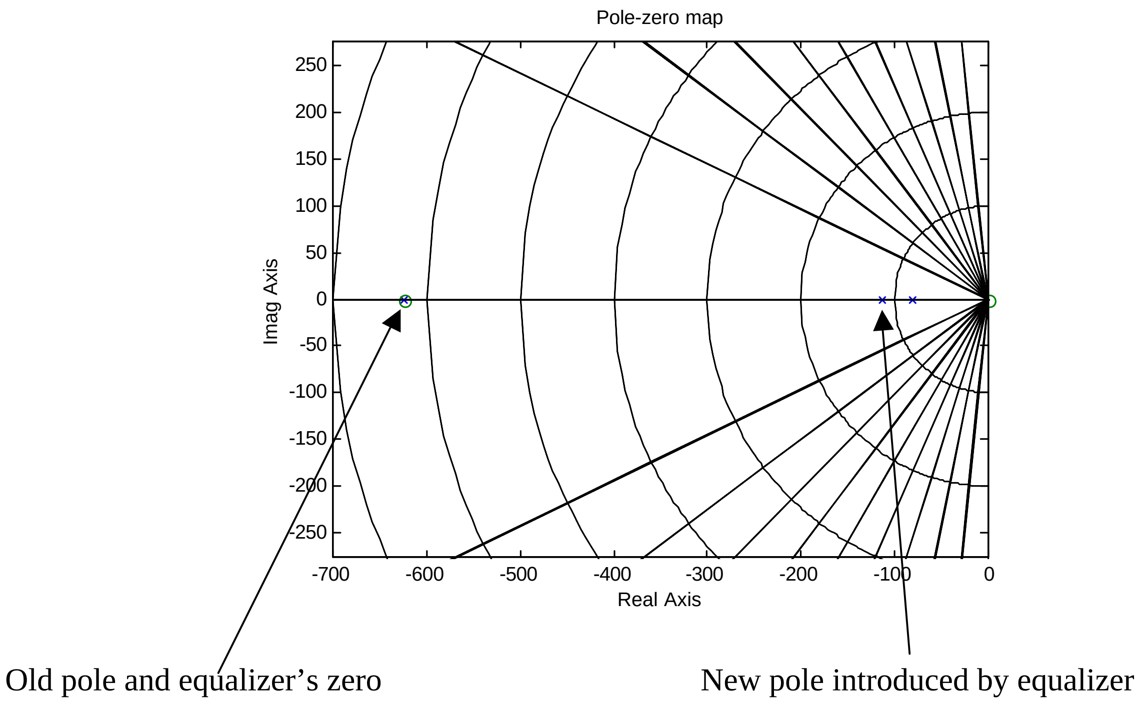

Figure 25. Pole-zero map for equalized system.

Equalizer produces new pole at frequency of 12.88 Hz and it is shown at right side. Q_{tc} and f_c for equalized system is therefore:

$$f_c = \frac{1}{2p\sqrt{\frac{1}{80,47}\cdot\frac{1}{80,95}}} = 12,85\,\mathrm{Hz} \tag{4.13}$$

$$Q_{tc} = \frac{\sqrt{\frac{1}{80,47}\cdot\frac{1}{80,95}}}{\frac{1}{80,47}+\frac{1}{80,95}} = 0,5 \tag{4.14}$$

The equalized system is therefore a critically damped one. Step response lenghtens a bit, but not excessively.

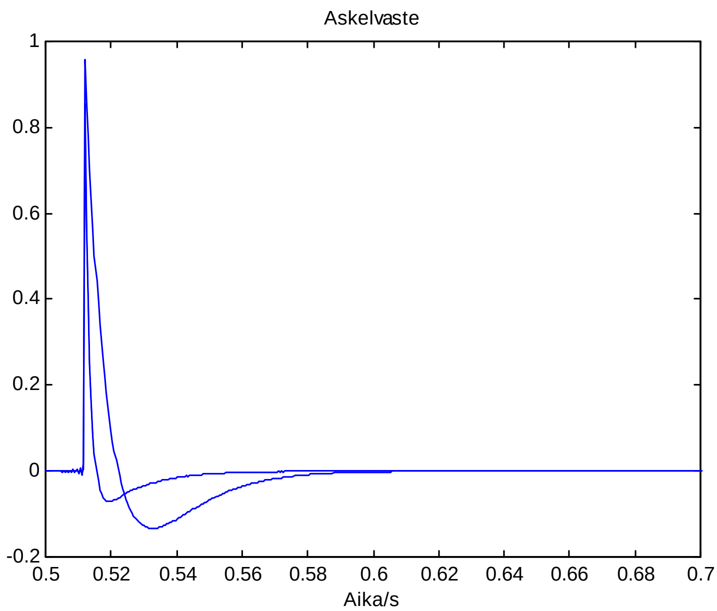

Figure 26. The step response of equalized system.

Perhaps a more practical transient signal is sine burst signal which is shown below, and response of equalized system to it.

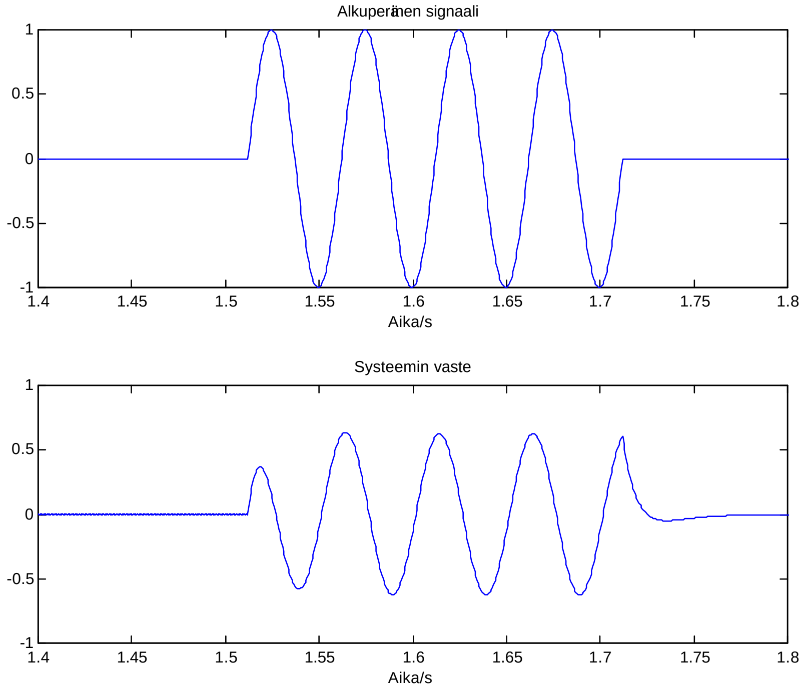

Figure 27. Tone burst response of equalized system.

Tone burst response is also good. No ringing is evident.

4.2 Equalization of the ”high-Q” design (Q_{tc}>0.5)

For this example, I chose Infinity Beta 15X driver with 60 litres closed box. I have seen two such articles. In this case, Q_{tc} is larger than 0,5 so we must use Linkwitz-transform circuit for equalization.

Infinity Beta 15X has following parameters:

5 ACKNOWLEDGEMENTS

Author wishes to thank Juha Hartikainen and Tommi Prami for motivation to write this document and Mr. Siegfried Linkwitz for his systematic research and excellent articles in this area.

6 REFERENCES

[1] S. H. Linkwitz, ”Loudspeaker system design”, *Wireless World*, December 1978, p. 80

[2] J. Karki, ”Analysis of the Sallen-Key Architecture”, Texas Instruments Incorporated application note SLOA024A, http://www-s.ti.com/sc/psheets/sloa024a/sloa024a.pdf, July 1999

[3] P. Tuomela, *Rakenna HIFI-kaiuttimia*, (Tecnopress, 1992(?)), pp. 98-111.

[4] Bagend, ”Guide to ELF Systems, A New Era in Bass Reproduction”, Modular sound systems Inc., 1997